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Auslander–Reiten Sequences over Artinian Rings

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Presently two classes of artinian rings are known, over which there exist Auslander–Reiten sequences: artin algebras and rings of finite representation type. To be more precise, Auslander and Reiten have proved the existence of an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ consisting of finitely generated modules for each finitely generated, indecomposable, non-projective module C (resp. for each finitely generated, indecomposable, non-injective module A) over such a ring. Moreover, these sequences are even Auslander–Reiten sequences in the category of all modules, i.e., the characteristic factorization properties are valid in the whole module category. Little seems to be known about further classes of artinian rings which possess these very properties.

In this paper we indicate an example of an artinian ring R which is neither an artin algebra nor of finite representation type, but which admits Auslander–Reiten sequences in the category mod R. However, contrary to the situation over the first mentioned rings, some of these sequences are not Auslander–Reiten sequences in the category Mod R of all modules. The presentation of this example takes up the major part of this work. The ring we are dealing with is the triangular matrix ring

$$R = \begin{pmatrix} F & {}_F N_F \\ 0 & F \end{pmatrix},$$

where F denotes the field of formal Laurent series in one variable over a field k of characteristic 0 and ${}_F N_F$ a bimodule over F, which as a left F-vector space has a basis x, y and as a right vector space is defined by $x\alpha := \alpha x$ and $y\alpha := \alpha y + \alpha' x$, α' standing for the usual derivative of $\alpha \in F$. The representation theory of such rings has been extensively studied by Ringel [8]. Using his results we shall show under Section 2 that Auslander–Reiten sequences exist in mod R. Caused by the specific nature of R, systems of differential equations over F play a crucial role in our argumentation.

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Therefore we need to show that the solution space of a homogeneous system of linear differential equations over F is finite-dimensional over k and we also need a criterion for the solvability of certain systems of non-homogeneous equations. These technical tools will be developed under Section 3. Section 1 is of a general nature. First we show that for a finitely presented, non-projective module C with a local endomorphism ring over a semiperfect ring R there exists an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in $\operatorname{mod} R$, if and only if $(\operatorname{Tr} C)^0$ contains a finitely presented pure submodule A with a local endomorphism ring. ($(\operatorname{Tr} C)^0$ is defined below.) Then we study artinian rings R admitting Auslander–Reiten sequences $0 \to A \to B \to C \to 0$ in $\operatorname{mod} R$ for all simple non-projective modules C. In the end several results are stated which allow us to derive the existence of Auslander–Reiten sequences from given ones.

In this paper, all rings have an identity and all modules are unitary. For some ring R, $\operatorname{Mod} R$ ($R \operatorname{Mod}$) will denote the category of all right (left) modules and $\operatorname{mod} R$ ($R \operatorname{mod}$) the category of finitely presented right (left) modules over R. Instead of $\operatorname{Hom}_R(M_R, N_R)$ we shall write (M_R, N_R) or (M, N). $J(R)$ denotes the Jacobson radical of R, $\operatorname{Soc} M$ the socle of a module M; $\operatorname{Soc}^i M$, $i \geqslant 1$, is defined inductively by $\operatorname{Soc}^1 M = \operatorname{Soc} M$ and $\operatorname{Soc}^{i+1} M/\operatorname{Soc}^i M = \operatorname{Soc}(M/\operatorname{Soc}^i M)$. For a finitely presented module M without projective direct summands $\neq 0$ over a semiperfect ring, $\operatorname{Tr} M$ denotes the transposed module. If M_R is a module with a local endomorphism ring S and ${}_S U$ an injective hull of ${}_S(S/J(S))$, then we put $M^0 := ({}_S M, {}_S U)$. Finally we recall the definition of an Auslander–Reiten sequence (AR-sequence for short). A non-split exact sequence $0 \to A \to^f B \to^g C \to 0$ in $\operatorname{Mod} R$ ($\operatorname{mod} R$) is called an AR-sequence in $\operatorname{Mod} R$ ($\operatorname{mod} R$), if it has the following two properties: For each module M in $\operatorname{Mod} R$ ($\operatorname{mod} R$) and each $s \in (M, C)$ which is not a split epimorphism there exists $s' \in (M, B)$ with $gs' = s$, and for each $N \in \operatorname{Mod} R$ ($\operatorname{mod} R$) and each $t \in (A, N)$ which is not a split monomorphism there exists $t \in (B, N)$ with $t'f = t$. A very general existence theorem is attributed to Auslander. We quote it, because it forms the background of most results in this paper.

THEOREM (Auslander [3, Theorem 3.9]). *Let R be semiperfect. For each non-projective $C \in \operatorname{mod} R$ with a local endomorphism ring there is an Auslander–Reiten sequence* $0 \to (\operatorname{Tr} C)^0 \to B \to C \to 0$ *in* $\operatorname{Mod} R$.

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This section contains a number of general statements centred around the existence of Auslander–Reiten sequences in $\operatorname{mod} R$. They complement and generalize related results in our paper [10]. To begin with, we derive an existence theorem for Auslander–Reiten sequences in $\operatorname{mod} R$ from Auslander's theorem. It is illustrated in a striking way by the example in the next section.

THEOREM 1. *Let R be a semiperfect ring and C_R a finitely presented, non-projective module with a local endomorphism ring. There exists an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in* $\operatorname{mod} R$ *if and only if* $(\operatorname{Tr} C)^0$ *contains a finitely presented, pure submodule with a local endomorphism ring. Up to isomorphism, the latter is uniquely determined.*

Proof. We have shown in [10, Proposition 3], that if $0 \to A \to B \to C \to 0$ is an AR-sequence in mod R, then there exists a pure embedding $A \to (\operatorname{Tr} C)^0$.

To settle the converse implication, we assume that there exists a finitely presented module A with a local endomorphism ring and a pure monomorphism i: $A \to (\operatorname{Tr} C)^0$. By Auslander's theorem there exists an AR-sequence $0 \to (\operatorname{Tr} C)^0 \to^{f'} B' \to^{g'} C \to 0$ in Mod R. We may assume that i is not an isomorphism. Now we denote by p: $(\operatorname{Tr} C)^0 \to K$ the cokernel of i. Because p is not a split monomorphism, there is a map $q \in (B', K)$ with $qf' = p$; let j: $B \to B'$ denote the kernel of q. These maps induce in an obvious way the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\operatorname{Tr} C)^0 \arrow[r, "f'"] \arrow[d, "p"] & B' \arrow[r, "g'"] \arrow[d, "q"] & C \arrow[r] & 0 \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The proof is finished by showing that the first row is an AR-sequence in mod R. It does not split, because the middle row does not. Since A and C have local endomorphism rings, it is sufficient to prove that g is right almost split in mod R. Let $M \in \operatorname{mod} R$ and $g \in (M, C)$ be a morphism which is not a split epimorphism. g' being right almost split in Mod R, there is $h_1 \in (M, B')$ such that $g'h_1 = h$. By purity of i there exists

$h_2 \in (M, (\mathrm{Tr}\ C)^0)$ such that $qh_1 = ph_2 = qf'h_2$, hence there is $h_3 \in (M, B)$ satisfying $jh_3 = h_1 - f'h_2$ and we have $h = g'h_1 = g'(jh_3 + f'h_2) = g'jh_3 = gh_3$. Because AR-sequences are uniquely determined up to isomorphism, A has this property, also.

The dual question of the existence of an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in mod R for given A can be answerd similarly.

THEOREM 2. *Let R be semiperfect and A_R a finitely presented, non-injective module with a local endomorphism ring.*

(1) *There exists an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in* mod R *if and only if there is a finitely presented, non-projective module ${}_RX$ with a local endomorphism ring and a pure embedding $A \to X^0$.*

(2) *Furthermore, if A is not projective, then there exists an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in* mod R *if and only if A^0 contains a finitely presented, non-projective pure submodule with a local endomorphism ring.*

Proof. (1) It is an easy consequence of Theorem 1.

(2) In case A is not projective then by [10, Satz 4] there exists an AR-sequence $0 \to A \to B \to C \to 0$ in mod R iff there exists an AR-sequence $0 \to X \to Y \to \mathrm{Tr}\ A \to 0$ in R mod where X is not projective. Hence our assertion likewise follows from Theorem 1.

By including further conditions we can slightly improve the statements of the last theorem.

Remark 3. We adopt the assumptions of Theorem 2; furthermore, we suppose that A_R is algebraically compact or ${}_RR$ is noetherian.

(1) If there is a finitely presented module ${}_RX$ with a local endomorphism ring and a pure embedding $A \to X^0$, then X is not projective.

(2) We assume additionally that ${}_RR$ is algebraically compact. If A^0 contains a finitely presented pure submodule ${}_RY$ with a local endomorphism ring, then Y is not projective.

Proof. (1) If X is projective then X^0 is injective and, by our assumptions, the map $A \to X^0$ splits, a contradiction.

(2) Now we suppose that ${}_RR$ is algebraically compact and ${}_RY$ projective. Then Y is algebraically compact too, hence a direct summand of A^0. Because A^0 is indecomposable we may infer $Y = A^0$. Let $S = \mathrm{End}(A_R)$, ${}_SV$ an injective hull of $S/J(S)$, $T = \mathrm{End}({}_RA^0) = \mathrm{End}({}_SV)$, and U_T an injective hull of V_T. Since the canonical embedding $A \to (A^0_T, U_T)$ is pure, we arrive at a contradiction in the same way as in (1).

Next we show a criterion for the existence of Auslander–Reiten sequences in mod R ending with simple modules. The dual problem, concerning the existence of Auslander–Reiten sequences, which begin with simple modules, was treated in [10, Folgerung 10].

THEOREM 4. *For a right artinian ring R the following statements are equivalent*:

(1) *For each simple non-projective module C_R there exists an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in* mod R.

(2) *R is a right Morita ring, i.e., the minimal cogenerator Q_R is finitely generated, and $S = \operatorname{End}(Q_R)$ is artinian on either side.*

Furthermore, if conditions (1) *and* (2) *hold, then each Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in* mod R *with simple C is even an Auslander–Reiten sequence in* mod R *and $A \cong (\operatorname{Tr} C)^0 \cong (\operatorname{Tr}((C, Q)^0), Q)$.*

Proof. (1) $\Rightarrow$ (2) To show that Q_R is finitely generated we suppose that some indecomposable injective right module is not finitely generated. Then there exists an indecomposable injective module E_R such that $\operatorname{Soc}^2 E$ is not finitely generated. Hence there is a submodule M of $\operatorname{Soc}^2 E$ the length of which is larger than the lengths of all middle terms of AR-sequences $0 \to A \to B \to C \to 0$ in mod R, C running through the set of all simple factor modules of M. Now let h: $M \to C$ be an epimorphism onto a simple module C and $0 \to A \to^f B \to^g C \to 0$ an AR-sequence in mod R. Because h does not split, there exists a morphism $h' \in (M, C)$ with $gh' = h$. h' cannot be a monomorphism by choice of M, hence factorizes over the canonical map p: $M \to M/\operatorname{Soc} M$. Because h also factorizes over p, g is a split epimorphism, a contradiction.

Now, since Q_R is finitely generated, S is left artinian and ${}_SQ_R$ establishes a Morita duality between mod R and S mod. Applying this duality to AR-sequences in mod R which end with simple modules, we see that for each simple, non-injective left S-module there exists an AR-sequence beginning with it. Hence, by [10, Folgerung 10], $J(S)_S$ is finitely generated and S is right artinian.

(2) $\Rightarrow$ (1) If C_R is simple, non-projective, then ${}_S(C_R, Q_R)$ is simple, non-injective. Therefore, by [10, Folgerung 10], there exists an AR-sequence $0 \to {}_S(C, Q) \to {}_SX \to \operatorname{Tr}((C, Q)^0) \to 0$, and the dual $0 \to (\operatorname{Tr}((C, Q)^0), Q) \to ({}_SX, {}_SQ) \to ({}_S(C, Q), {}_SQ) \cong C \to 0$ is an AR-sequence in mod R.

Finally we assume (1) and (2) and show that in each AR-sequence E: $0 \to A' \to^{f'} B' \to^{g'} C \to 0$ in Mod R with simple C the first term A' is finitely generated. Let us call an exact sequence $0 \to X \to Y \to Z \to 0$ in Mod R copure if for each finitely generated module M_R the sequence

$0 \to (Z, M) \to (Y, M) \to (X, M) \to 0$ is exact, too. If we suppose that A' is not finitely generated then E is copure, and because f' is obviously an essential monomorphism the following lemma yields a contradiction.

LEMMA 5. *Let R be a right artinian, right Morita ring, Q_R the minimal cogenerator, and $S = \operatorname{End}(Q_R)$ (recall that ${}_SS$ is artinian).*

(1) *An exact sequence E: $0 \to X \to Y \to Z \to 0$ is copure if and only if the dual (E, Q): $0 \to {}_S(Z, Q) \to {}_S(Y, Q) \to {}_S(X, Q) \to 0$ is pure.*

(2) *S is right artinian if and only if the dual (α, Q): ${}_S(Y, Q) \to {}_S(X, Q)$ of each essential monomorphism α: $X \to Y$ is a superfluous epimorphism.*

(3) *If S_S is artinian and $0 \to X \to^{\alpha} Y \to Y \to 0$ a copure exact sequence with essential α, then α is an isomorphism.*

Proof. (1) It follows from the fact that ${}_SQ_R$ establishes a duality $\operatorname{mod} R \to S \operatorname{mod}$ and that for all modules M_R and ${}_SN$ there exists a canonical isomorphism $({}_SM, {}_S(N_R, Q_R)) \cong (N_R, ({}_SM, {}_SQ)_R)$.

(2) First we assume that S_S is artinian and $0 \to X \to^{\alpha} Y \to^{\beta} Z \to 0$ an exact sequence with essential α. We have to show that $\varphi\beta \in J(S) \cdot (Y, Q)$ for all $\varphi \in (Z, Q)$. Because the kernel of $\varphi\beta$ is essential, it factorizes over the canonical map π: $Y \to Y/\operatorname{Soc} Y$; i.e., there exists $\gamma \in (Y/\operatorname{Soc} Y, Q)$ such that $\varphi\beta = \gamma\pi$. Let us choose an embedding j: $Y \to Q^{(I)}$ where I denotes some index set. Since $\operatorname{Soc} Y = Y \cap \operatorname{Soc} Q^{(I)}$, j induces a monomorphism l: $Y/\operatorname{Soc} Y \to Q^{(I)}/\operatorname{Soc} Q^{(I)}$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d, "j"'] & Y/\operatorname{Soc} Y \arrow[r, "\gamma"] \arrow[d, "l"] & Q \\ Q^{(I)} \arrow[r, "\nu = \mathrm{can}"'] & Q^{(I)}/\operatorname{Soc} Q^{(I)} \arrow[ur, dashed, "\delta"'] & \end{tikzcd}$$

commutes. Hence there exists a homomorphism δ with $\delta l = \gamma$. Obviously the components δ_i, $i \in I$, of $\delta\nu$ are elements of $J(S)$. Because S_S is noetherian, the right ideal generated by $(\delta_i)_{i \in I}$ is generated by a finite subfamily $(\delta_i)_{i \in K}$. This means that there is a morphism λ: $Q^{(I)} \to Q^K$ satisfying $\delta\nu = (\delta_i)_{i \in K} \circ \lambda$, hence $\varphi\beta = \gamma\pi = \delta l \pi = \delta\nu j = (\delta_i)_{i \in K} \circ \lambda j \in J(S) \cdot (Y, Q)$. Conversely we suppose that the duals of essential monomorphisms are superfluous epimorphisms and choose a generating family $(s_i)_{i \in I}$ of $J(S)_S$. Because the kernels of all s_i are essential, the kernel of the induced map $(s_i)_{i \in I}$: $Q^{(I)} \to Q$ is essential, too. By hypothesis the image of the dual map ${}_SS \to {}_SS^I$ is superfluous; i.e., $(s_i)_{i \in I} \in J(S) \cdot S^I$, hence there exist $t_1, ..., t_n \in J(S)$ and $p_1, ..., p_n \in S^I$ with $(s_i)_{i \in I} = \sum_{k=1}^{n} t_k p_k$. Obviously the $t_1, ..., t_n$ generate $J(S)_S$.

(3) Let S_S be artinian and $0 \to X \to^{\alpha} Y \to^{\beta} Z \to 0$ a copure exact sequence with essential α. Then the dual $0 \to (Z, Q) \to^{\tilde{\beta} = (\beta, Q)} (Y, Q) \to^{\tilde{\alpha} = (\alpha, Q)} (X, Q) \to 0$ is pure and $\tilde{\alpha}$ a superfluous epimorphism, hence $\operatorname{Im} \tilde{\beta} = (J(S) \cdot (Y, Q)) \cap \operatorname{Im} \tilde{\beta} = J(S) \cdot \operatorname{Im} \tilde{\beta}$. Since $J(S)$ is nilpotent, we may infer $\operatorname{Im} \tilde{\beta} = 0$ and $Z = 0$.

COROLLARY 6. *Let R be a quasi-Frobenius ring. For each simple non-projective module C_R there exists an AR-sequence $0 \to A \to B \to C \to 0$ in* Mod R *with finitely generated A and* $A \cong (\operatorname{Tr} C)^0 \cong (\operatorname{Tr}((C^*)^0))^*$ (* *denotes the dual with respect to R).*

Proof. This follows immediately from Theorem 4.

Theorem 4 may also be applied in order to improve a characterization of pure semisimple rings which is due to Brune [4, Corollary 1 of Theorem 2]. He has shown that a right artinian ring R is right pure semisimple if and only if the following conditions hold:

(i) Each simple right R-module admits generalized right almost split sequences.

(ii) The injective hulls of all simple right R-modules are finitely generated.

Since in virtue of (i) for each simple non-projective module C_R there exists an AR-sequence $0 \to A \to B \to C \to 0$ in mod R, Theorem 4 guarantees that (i) implies (ii). As a consequence R is right pure semisimple if and only if every simple right R-module admits generalized right almost split sequences.

In the remaining part of this section we shall indicate various methods for the construction of Auslander–Reiten sequences from given ones. First we assume that we have a self-duality D: mod $R \to R$ mod with inverse D: R mod $\to$ mod R. If $0 \to A \to B \to C \to 0$ is an AR-sequence in mod R, then $0 \to DC \to DB \to DA \to 0$ is an AR-sequence in R mod. Furthermore, if A is not projective, then there exists an AR-sequence $0 \to \operatorname{Tr} C \to \tilde{B} \to \operatorname{Tr} A \to 0$ in R mod [10, Satz 4], and the dual $0 \to D \operatorname{Tr} A \to D\tilde{B} \to D \operatorname{Tr} C \to 0$ is again an AR-sequence in mod R. Similarly, if C is not injective, then there is an AR-sequence $0 \to \operatorname{Tr} DA \to \widetilde{DB} \to \operatorname{Tr} DC \to 0$ in mod R.

Now let R be a quasi-Frobenius ring, $0 \to A \to^{f} B \to^{g} C \to 0$ an Auslander–Reiten sequence in mod R, and a_1: $A_1 \to A$, c_1: $C_1 \to C$ projective hulls. It is well known that the kernels $\Omega A := \operatorname{Ker}(a_1)$ and $\Omega C := \operatorname{Ker}(c_1)$ are indecomposable and that the exact sequences $0 \to \Omega A \to A_1 \to A \to 0$ and $0 \to \Omega C \to C_1 \to C \to 0$ may be embedded into a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega A \arrow[r, "\tilde{f}"] \arrow[d, "a_2"'] & \tilde{B} \arrow[r, "\tilde{g}"] \arrow[d, "b_2"'] & \Omega C \arrow[r] \arrow[d, "c_2"'] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "f_1"'] \arrow[d, "a_1"'] & B_1 \arrow[r, "g_1"'] \arrow[d, "b_1"'] & C_1 \arrow[r] \arrow[d, "c_1"'] & 0 \\
0 \arrow[r] & A \arrow[r, "f"'] \arrow[d] & B \arrow[r, "g"'] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

THEOREM 7. *The upper row* E: $0 \to \Omega A \to^{\tilde{f}} \tilde{B} \to^{\tilde{g}} \Omega C \to 0$ *is an Auslander–Reiten sequence in* mod R.

Note that in the special case when R is a quasi-Frobenius artin algebra, Theorem 7 has been proved formerly by Auslander and Reiten [2, Proposition 5.1].

Proof. First we show that E does not split. If we suppose the contrary there exists s: $\Omega C \to \tilde{B}$ with $\tilde{g}s = 1$. Since B_1 is injective, s may be lifted to s_1: $C_1 \to B_1$ with $s_1 c_2 = b_2 s$, and s_1 induces s_2: $C \to B$ such that $s_2 c_1 = b_1 s_1$:

$$\begin{tikzcd}
\tilde{B} \arrow[r, shift left, "\tilde{g}"] \arrow[d, "b_2"'] & \Omega C \arrow[l, shift left, dashed, "s"] \arrow[d, "c_2"] \\
B_1 \arrow[r, shift left, "g_1"] \arrow[d, "b_1"'] & C_1 \arrow[l, shift left, dashed, "s_1"] \arrow[d, "c_1"] \\
B \arrow[r, shift left, "g"] & C \arrow[l, shift left, dashed, "s_2"]
\end{tikzcd}$$

Because $(1 - g_1 s_1)c_2 = 0$, there exists t: $C \to B_1$ with $g_1 s_1 + t c_1 = 1$ and from $c_1 = c_1(g_1 s_1 + t c_1) = (g s_2 + c_1 t) c_1$ we may infer $g s_2 + c_1 t = 1$. Because $\mathrm{End}(C)$ is local and $g s_2$ and $c_1 t$ cannot be isomorphisms, they are elements of $J(\mathrm{End}(C))$, a contradiction.

Next we have to check that $\tilde{g}$ is right almost split in mod R. Let $M \in \mathrm{mod}\, R$ be indecomposable, non-projective, h: $M \to \Omega C$ a morphism which is not a split epimorphism, i: $M \to M_1$ an injective hull, and p: $M_1 \to \Omega^{-1} M$ the cokernel of i. Then $\Omega^{-1} M$ is indecomposable and p a projective hull of $\Omega^{-1} M$. Since C_1 is injective, there exists h_1: $M_1 \to C_1$ with $h_1 i = c_2 h$ and h_1 induces h_2: $\Omega^{-1} M \to C$ with $h_2 p = c_1 h_1$:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\tilde{B} \arrow[r, "\tilde{g}"] \arrow[d, "b_2"'] & \Omega C \arrow[d, "c_2"'] & M \arrow[l, "h"'] \arrow[d, "i"] \arrow[ll, dashed, bend left, "t_2"] \\
B_1 \arrow[r, "g_1"] \arrow[d, "b_1"'] & C_1 \arrow[d, "c_1"'] & M_1 \arrow[l, "h_1"'] \arrow[d, "p"] \arrow[ll, dashed, bend left, "t_1"] \\
B \arrow[r, "g"] \arrow[d] & C \arrow[d] & \Omega^{-1}M \arrow[l, "h_2"'] \arrow[d] \arrow[ll, dashed, bend left, "t"] \\
0 & 0 & 0
\end{tikzcd}$$

Clearly h_2 is not an isomorphism, hence not a split epimorphism. Hence, g being right almost split, there exists $t\colon \Omega^{-1}M \to B$ with $gt = h_2$. Furthermore, M_1 being projective, there is $t_1\colon M_1 \to B_1$ with $b_1 t_1 = tp$ and t_1 induces $t_2\colon M \to \tilde{B}$ such that $b_2 t_2 = t_1 i$. The equation $c_1(h_1 - g_1 t_1) = 0$ implies the existence of $s\colon M_1 \to \Omega C$ with $h_1 = g_1 t_1 + c_2 s$ and since $c_2(h - \tilde{g}t_2 - si) = (h_1 - g_1 t_1 - c_2 s)i = 0$ we have $h = \tilde{g}t_2 + si$. Again using the projectivity of M_1 we find $\tilde{s}\colon M_1 \to \tilde{B}$ with $\tilde{g}\tilde{s} = s$ and thereby the desired factorization $h = \tilde{g}(t_2 + \tilde{s}i)$.

There is an obvious dual construction. If $a_1\colon A \to A_1$ and $c_1\colon C \to C_1$ denote injective hulls, $a_2\colon A_1 \to \Omega^{-1}A$ and $c_2\colon C_1 \to \Omega^{-1}C$ the cokernels of a_1 and a_2, respectively, then we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "a_1"'] & B \arrow[r, "g"] \arrow[d, "b_1"'] & C \arrow[r] \arrow[d, "c_1"'] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "a_2"'] & B_1 \arrow[r] \arrow[d, "b_2"'] & C_1 \arrow[r] \arrow[d, "c_2"'] & 0 \\
0 \arrow[r] & \Omega^{-1}A \arrow[r] \arrow[d] & \tilde{B} \arrow[r] \arrow[d] & \Omega^{-1}C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and the dual of the preceding proof shows that the lower row is an Auslander–Reiten sequence in mod R.

COROLLARY 8. *Let R be a quasi-Frobenius ring, C_R a simple, non-projective module, and M an injective hull of C. Then there are isomorphisms $\Omega(MJ) \cong (\operatorname{Tr} C)^0 \cong (\operatorname{Tr}((C^*)^0))^*$ and $\Omega^{-1}(M/C) \cong \operatorname{Tr}((M/MJ)^0)$.*

Proof. It is well known [1, Proposition 4.11] that $0 \to MJ \to^{(\pi,\tau)} MJ/C \oplus M \to^{(-j,v)} M/C \to 0$ is an AR-sequence in mod R, where π: $MJ \to MJ/C$, τ: $MJ \to M$, j: $MJ/C \to M/C$, and v: $M \to M/C$ are the natural maps. Theorem 7 yields an AR-sequence $0 \to \Omega(MJ) \to Z \to \Omega(M/C) \cong C \to 0$ in mod R, and Corollary 6 the isomorphisms $\Omega(MJ) \cong (\operatorname{Tr} C)^0 \cong (\operatorname{Tr}((C^*)^0))^*$. The second statement is proved similarly.

THEOREM 9. *Let R be right artinian, I a two-sided ideal of R, and $0 \to A \to^f B \to^g \to C \to 0$ an Auslander–Reiten sequence in* mod R.

(1) *If $CI = 0$ and C is not projective as an R/I-module, then there exists an Auslander–Reiten sequence $0 \to A_1 \to B_1 \to C \to 0$ in* mod R/I.

(2) *If $AI = 0$ and A is not injective as an R/I-module, then there is an Auslander–Reiten sequence $0 \to A \to B_1 \to C_1 \to 0$ in* mod R/I.

Proof. (1) We put $X' := (R/I_R, X_R)$ for a module X_R. Because C is not R/I-projective, the induced sequence $0 \to A' \to^{f'} B' \to^{g'} C' = C \to 0$ is exact, and obviously it does not split. Let $A' = A_1 \oplus \cdots \oplus A_n$ be a decomposition into indecomposables. Then for some summand, say A_1, the projection π_1: $A' \to A_1$ is not factorizable over f'. We shall show that the lower row in the pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle v_1} & & \| & & \\
0 & \longrightarrow & A_1 & \xrightarrow[f_1]{} & B_1 & \xrightarrow[g_1]{} & C & \longrightarrow & 0
\end{array}$$

is an AR-sequence in mod R. By choice of π_1 it does not split. Since A_1 and C have local endomorphism rings, it is sufficient to show that g_1 is right almost split in mod R/I. Let $M \in \operatorname{mod} R/I$ and h: $M \to C$ not a split epimorphism. Then there is h_1: $M \to B$ with $gh_1 = h$ and $h_1(M) \subset B'$ implies $h = g'h_1 = g_1(v_1 h_1)$.

(2) It is proved dually.

PROPOSITION 10. *Suppose that R is right artinian and A_R a finitely generated indecomposable module, which is neither injective nor projective.*

Provided there exist Auslander–Reiten sequences $0 \to A \to^{f} B \to C \to 0$ *and* $0 \to U \to V \to A \to 0$ *in* Mod R *consisting of finitely generated modules, then for each indecomposable non-projective direct summand* B_1 *of* B *there exists an Auslander–Reiten sequence* $0 \to X \to Y \to B_1 \to 0$ *in* Mod R *with finitely generated* X.

Proof. Let $0 \to X \to^{h} Y \to B_1 \to 0$ be an AR-sequence in Mod R. Because the map $f_1\colon A \to B_1$, induced by f, is irreducible in Mod R, A is isomorphic to a direct summand A_1 of Y. The map $h_1\colon X \to A_1$ which is induced by h is irreducible in Mod R; hence X is isomorphic to a direct summand of V and therefore finitely generated.

COROLLARY 11. *We assume that* R *is artinian (on either side) and a right Morita ring, furthermore, that the endomorphism ring of the minimal cogenerator is artinian, also. If* C_R *is a simple module which is neither projective nor injective and* $0 \to A \to B \to C \to 0$ *an Auslander–Reiten sequence in* mod R, *then for each indecomposable, non-projective direct summand* B_1 *of* B *there exists an Auslander–Reiten sequence* $0 \to X \to Y \to B_1 \to 0$ *in* Mod R *with finitely generated* X.

Proof. By Theorem 4 the sequence $0 \to A \to B \to C \to 0$ is even an AR-sequence in Mod R. Because there exists an AR-sequence $0 \to C \to D \to E \to 0$ in Mod R with finitely generated E [10, Folgerung 10] the corollary follows from Proposition 9.

PROPOSITION 12. *Let* R *be artinian,* A_R *an indecomposable projective, non-injective module, and* $0 \to A \to B \to C \to 0$ *an Auslander–Reiten sequence in* mod R. *Then for each indecomposable, non-projective direct summand* B_1 *of* B *there exists an Auslander–Reiten sequence* $0 \to X \to Y \to B_1 \to 0$ *in* Mod R *consisting of finitely generated modules.*

Proof. First we note that the map $f_1\colon A \to B_1$, induced by f, is irreducible in Mod R. Hence, if $0 \to X \to^{h} Y \to B_1 \to 0$ is an AR-sequence in Mod R, then A is isomorphic to a direct summand A_1 of Y and the map $h_1\colon X \to A_1$ induced by h is irreducible in Mod R. Because A_1 is projective, X is isomorphic to a direct summand of $A_1 J$, hence finitely generated.

2

Now we shall present an example illustrating the situation described by Theorem 1. We shall construct an artinian ring R which is neither an artin algebra nor of finite representation type such that for each finitely generated, indecomposable, non-projective module C_R there exists an

Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in $\operatorname{mod} R$ and for each finitely generated, indecomposable, non-injective module A_R an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in mod R. The most remarkable feature of our example lies in the fact that some of these sequences are not Auslander–Reiten sequences in Mod R.

We begin by exposing the construction of R and recalling some of the properties of the category mod R. Let k be some field of characteristic 0, $F := k((X))$ the field of formal Laurent series in the variable X over k, i.e., the quotient field of the power series ring $k[[X]]$, and ${}_FN$ a left vector space over F with basis x, y. By means of the natural derivation $F \to F$, $\alpha \mapsto \alpha'$, the space N can in addition be made a right F-vector space by $x\alpha := \alpha x$ and $y\alpha := \alpha y + \alpha' x$. Obviously, x and y form a right basis of N, also. We define R as the triangular matrix ring $\left(\begin{smallmatrix} F & N \\ 0 & F \end{smallmatrix}\right)$. R is an F–F-bimodule via $F \to R$, $\alpha \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)$, and the elements $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$ form a left and a right basis of R over F; mostly we shall write x resp. y instead of $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$ resp. $\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$. It is well-known that R is artinian and hereditary on either side; R is not an artin algebra, because its center is equal to $k \cdot 1$. Furthermore, since the map $(N_F, F_F) \to ({}_FN, {}_FF)$, $\varphi \mapsto \tilde{\varphi}$, where $\tilde{\varphi}(x) := \varphi(x)$, $\tilde{\varphi}(y) := \varphi(x)' + \varphi(y)$, is an F–F-bimodule isomorphism, there exists a weakly symmetric duality D: $\operatorname{mod} R \to R \operatorname{mod}$, and $D\operatorname{Tr}$ resp. $\operatorname{Tr} D$ is isomorphic to the Coxeter functor C^+ resp. C^- belonging to R (see [9]).

R has infinite representation type. In fact, Ringel has proved the following classification of the finitely generated right R-modules according to their dimension types [8, Lemma 6.6]. Recall that the dimension type of M_R is the pair of natural numbers $(\dim Me_{1_F}, \dim Me_{2_F})$.

(i) For each pair (m, n) of numbers m, $n \in \{0, 1, 2, \ldots\}$ with $|m - n| = 1$ there exists precisely one indecomposable module of dimension type (m, n).

(ii) All the other finitely generated indecomposable modules have dimension type (n, n), $n \geqslant 1$. Their finite direct sums form a full exact abelian subcategory $\mathfrak{r}(N)$ which is the product $\mathfrak{m} \times \mathfrak{u}$ of two subcategories $\mathfrak{m}$ and $\mathfrak{u}$. The category $\mathfrak{m}$ is equivalent to the category of modules of finite length over the derivation polynomial ring $S = F[Y, ']$ over F in the variable Y (the elements of S are uniquely written in the form $\sum_{i=0}^{n} \alpha_i Y^i$ with $\alpha_i \in F$ and the multiplication is given by $Y\alpha := \alpha Y + \alpha'$); $\mathfrak{u}$ is a uniserial category of global dimension 1 with the only simple object $e_1 R/xR$.

Now we are in a position to prove the principal result of this section.

THEOREM 13. *For each finitely generated, indecomposable, non-projective* *right R-module C there exists an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in* mod R. *This is even an Auslander–Reiten sequence in* Mod R *if C belongs to* $\mathfrak{u}$ *or is of dimension type (m, n) with $|m-n|=1$, whereas it is not an Auslander–Reiten sequence in* Mod R *if C lies in* $\mathfrak{m}$.

Proof. We have to provide a particular proof for each of the different types of modules which are given by Ringel's classification.

(A) To begin with the most complicated case let C_R be an indecomposable in $\mathfrak{m}$. By Ringel's results [8, Theorem 7.4] there exists an indecomposable right S-module M of finite length such that $C = M \times M$ as F-spaces, the R-module structure of C being given by $(m_1, m_2)x := (0, m_1)$ and $(m_1, m_2)y := (0, m_1 Y)$ for m_1, $m_2 \in M$. In order to establish the assertion for C we shall show that $(\operatorname{Tr} C)^0$ contains a finitely generated, indecomposable, pure submodule and may then apply Theorem 1. In fact we shall specify a pure embedding $C \to (\operatorname{Tr} C)^0$. As a preparatory step into this direction we calculate the endomorphism ring T of Tr C. Because S is a simple principal ideal ring, there exists $f \in S$ with $M \cong S/fS$ [6, Theorem 3.11]; of course, we may assume $M = S/fS$. Let $f := \sum_{i=0}^{n} Y^i \alpha_i$ with $\alpha_n = 1$; then the residue classes 1, $\bar{Y}, \ldots, \bar{Y}^{n-1}$ modulo fS form an F-basis of M and the elements $c_i = (\bar{Y}^i, 0)$, $0 \leqslant i \leqslant n-1$, a generating system of C. We put

$$\mathfrak{B} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & -\alpha_0 \\ 1 & \ddots & & \vdots & -\alpha_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -\alpha_{n-1} \end{pmatrix} \in F^{n \times n}, \quad \mathfrak{C} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in F^{n \times n}$$

and $\mathfrak{A} = -\mathfrak{B}x + \mathfrak{C}y \in R^{n \times n}$. Then an easy computation shows that the sequence $0 \to e_2 R^n \to^a e_1 R^n \to^v C \to 0$ with $a((e_2 r_i)^t_{1 \leqslant i \leqslant n}) = \mathfrak{A} \cdot (e_2 r_i)^t_{1 \leqslant i \leqslant n}$ and $v((e_1 r_i)^t_{1 \leqslant i \leqslant n}) = \sum_{i=1}^{n} c_{i-1} r_i$ is exact. Dualization with respect to R yields the exact sequence

$$0 \longrightarrow Re_1^n \xrightarrow{a^*} Re_2^n \xrightarrow{w} \operatorname{Tr} C \longrightarrow 0, \tag{1}$$

where $((s_i e_1)_{1 \leqslant i \leqslant n})a^* = (s_i e_1)_{1 \leqslant i \leqslant n} \cdot \mathfrak{A}$ and w is the canonical map. For later use we note that C may be identified with

$$\left\{ \begin{pmatrix} \mathfrak{a}^t & x\mathfrak{b}^t + y\mathfrak{c}^t \\ 0 & 0 \end{pmatrix} \middle| \, \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in F^n \right\} \Big/ \{ \mathfrak{A}\mathfrak{d}^t \mid \mathfrak{d} \in F^n \}$$

and Tr C with

$$\left\{ \begin{pmatrix} 0 & \mathfrak{a}x + \mathfrak{b}y \\ 0 & \mathfrak{c} \end{pmatrix} \middle| \, \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in F^n \right\} \Big/ \{ \mathfrak{d}\mathfrak{A} \mid \mathfrak{d} \in F^n \};$$

in the following $\bar{\ }$ will denote residue classes modulo $\{\mathfrak{A}\mathfrak{b}' \mid \mathfrak{b} \in F^n\}$ or $\{\mathfrak{b}\mathfrak{A} \mid \mathfrak{b} \in F^n\}$. Because (1) is a minimal projective resolution of $\operatorname{Tr} C$, the endomorphism ring T of $\operatorname{Tr} C$ may be dentified with the set of matrices $\mathfrak{T} \in F^{n \times n}$ for which there is a matrix $\mathfrak{S} \in F^{n \times n}$ with $\mathfrak{A}\mathfrak{T} = \mathfrak{S}\mathfrak{A}$. It is easily seen that $\mathfrak{T}$ satisfies such an equation if and only if $\mathfrak{T}' = \mathfrak{B}\mathfrak{T} - \mathfrak{T}\mathfrak{B}$. Viewing $\mathfrak{T}$ as a column in F^{n^2}, the equation $\mathfrak{T}' = \mathfrak{B}\mathfrak{T} - \mathfrak{T}\mathfrak{B}$ is equivalent to an equation $\mathfrak{T}' = \mathfrak{C}\mathfrak{T}$ for a suitable matrix $\mathfrak{C} \in F^{n^2 \times n^2}$. Hence, by Theorem 15, T is finite-dimensional over the central subfield k. With this information we shall show that the injective hull of the simple module $(T/J(T))_T$ is isomorphic to $U_T := (T, k)_T$. Dualizing the canonical map $\pi\colon T \to T/J(T)$ with respect to k, we obtain the embedding $(\pi, 1)\colon (T/J(T), k) \to U_T$, the image of which coincides with the socle of U_T. Since the skew field $T/J(T)$ is finite-dimensional over k, it is a symmetric k-algebra [5, Proposition 9.8], hence there exists an isomorphism $(T/J(T), k)_T \cong (T/J(T))_T$, and we may conclude that $\operatorname{Soc} U_T$ is simple. Consequently $(\operatorname{Tr} C)^0 = (\operatorname{Tr} C_T, U_T) \cong (\operatorname{Tr} C, k)$. Next, in order to find an embedding $C \to (\operatorname{Tr} C)^0$ we shall calculate $C \otimes_R \operatorname{Tr} C$ and indicate a distinguished map $C \otimes_R \operatorname{Tr} C \to k$. By definition $C \otimes_R \operatorname{Tr} C \cong e_1 R^n \otimes_R Re_2^n/(e_1 R^n \otimes_R \operatorname{Im}(a^*) + \operatorname{Im}(a) \otimes_R Re_2^n)$, and obvious identifications yield $C \otimes_R \operatorname{Tr} C \cong N^{n \times n}/U$ with $U = F^{n \times n}\mathfrak{A} + \mathfrak{A}F^{n \times n}$. The R-balanced map $C \times \operatorname{Tr} C \to N^{n \times n}/U$ corresponding to this isomorphism is given by

$$\left(\overline{\begin{pmatrix} \mathfrak{a}' & x\mathfrak{b}' + y\mathfrak{c}' \\ 0 & 0 \end{pmatrix}}, \overline{\begin{pmatrix} 0 & \mathfrak{x}x + \mathfrak{y}y \\ 0 & \mathfrak{z} \end{pmatrix}}\right)$$
$$\mapsto (\mathfrak{a}'\mathfrak{x} + \mathfrak{b}'\mathfrak{y} + (\mathfrak{c}'\mathfrak{z})')x + (\mathfrak{a}'\mathfrak{y} + \mathfrak{c}'\mathfrak{z})\, y + U.$$

Now let $\rho\colon N^{n \times n} \to k$, $\mathfrak{X}x + \mathfrak{Y}y \mapsto (\operatorname{tr}(\mathfrak{X} + \mathfrak{Y}\mathfrak{B}))_{-1}$, where tr denotes the usual trace of a matrix and, for some $\xi \in F$, ξ_{-1} the coefficient of X^{-1}. Note that $(\xi')_{-1} = 0$ for all $\xi \in F$. For $\mathfrak{X} \in F^{n \times n}$ we have $\rho(\mathfrak{X}\mathfrak{A}) = \rho(-\mathfrak{X}\mathfrak{B}x + \mathfrak{X}y) = (\operatorname{tr}(-\mathfrak{X}\mathfrak{B} + \mathfrak{X}\mathfrak{B}))_{-1} = 0$ and $\rho(\mathfrak{A}\mathfrak{X}) = \rho((\mathfrak{X}' - \mathfrak{B}\mathfrak{X})x + \mathfrak{X}y) = (\operatorname{tr}(\mathfrak{X}' - \mathfrak{B}\mathfrak{X} + \mathfrak{X}\mathfrak{B}))_{-1} = (\operatorname{tr}(\mathfrak{X})')_{-1} - (\operatorname{tr}(\mathfrak{B}\mathfrak{X} - \mathfrak{X}\mathfrak{B}))_{-1} = 0$. Hence ρ induces a map $N^{n \times n}/U \to k$ which we denote again by ρ. By composition we arrive at the R-balanced map $\sigma\colon C \times \operatorname{Tr} C \to N^{n \times n}/U \to^{\rho} k$ given by

$$\sigma\left(\overline{\begin{pmatrix} \mathfrak{a}' & x\mathfrak{b}' + y\mathfrak{c}' \\ 0 & 0 \end{pmatrix}}, \overline{\begin{pmatrix} 0 & \mathfrak{x}x + \mathfrak{y}y \\ 0 & \mathfrak{z} \end{pmatrix}}\right)$$
$$= (\operatorname{tr}(\mathfrak{a}'\mathfrak{x} + \mathfrak{b}'\mathfrak{z} + (\mathfrak{c}'\mathfrak{z})' + (\mathfrak{a}'\mathfrak{y} + \mathfrak{c}'\mathfrak{z})\mathfrak{B}))_{-1}$$
$$= (\mathfrak{x}\mathfrak{a}' + \mathfrak{z}\mathfrak{b}' + \mathfrak{y}\mathfrak{B}\mathfrak{a}' + \mathfrak{z}\mathfrak{B}\mathfrak{c}')_{-1}.$$

Now, using the fact that the bilinear map $F \times F \to k$, $(\xi, \eta) \mapsto (\xi\eta)_{-1}$, is not degenerated (Section 3), it is not difficult to show that σ is not degenerated, also. In particular the induced map $\tilde{\sigma}\colon C \to (\operatorname{Tr} C, k)$ is injective. It remains to be demonstrated that the embedding $\tilde{\sigma}$ is pure. Given a system of equations

$$\sum_{i=1}^{r} \varphi_i a_{ji} = \tilde{\sigma}(d_j), \qquad 1 \leqslant j \leqslant r, \tag{2}$$

with $\varphi_i \in (\operatorname{Tr} C, k)$, $a_{ji} \in R$, $d_j \in C$, we have to find elements $g_i \in C$, $1 \leqslant i \leqslant r$, with

$$\sum_{i=1}^{n} g_i a_{ji} = d_j, \qquad 1 \leqslant j \leqslant r.$$

This problem is settled by transforming system (2) in such a way that Theorem 20 under Section 3 is applicable. Let

$$a_{ji} = \begin{pmatrix} \alpha_{ji} & \beta_{ji} x + \gamma_{ji} x \\ 0 & \delta_{ji} \end{pmatrix} \quad \text{and} \quad d_j = \overline{\begin{pmatrix} \mathfrak{a}_j^t & x\mathfrak{b}_j^t + y\mathfrak{c}_j^t \\ 0 & 0 \end{pmatrix}};$$

then (2) means that for each family of elements

$$p_j = \overline{\begin{pmatrix} 0 & \mathfrak{x}_j x + \mathfrak{y}_j y \\ 0 & \mathfrak{z}_j \end{pmatrix}} \in \operatorname{Tr} C, \qquad 1 \leqslant j \leqslant r,$$

the equations $\sum_{i=1}^{r} \varphi_i(a_{ji} p_j) = \tilde{\sigma}(d_j)(p_j)$, $1 \leqslant j \leqslant r$, hold, explicitly

$$\sum_{i=1}^{r} \varphi_i \left(\begin{pmatrix} 0 & [\alpha_{ji} \mathfrak{x}_j + \beta_{ji} \mathfrak{z}_j + \gamma_{ji} \mathfrak{z}_j' + (\alpha_{ji} \mathfrak{y}_j + \gamma_{ji} \mathfrak{z}_j) \mathfrak{B}] x \\ 0 & \delta_{ji} \mathfrak{z}_j \end{pmatrix} \right)$$
$$= (\mathfrak{x}_j \mathfrak{a}_j^t + \mathfrak{z}_j \mathfrak{b}_j^t + \mathfrak{y}_j \mathfrak{B} \mathfrak{a}_j^t + \mathfrak{z}_j \mathfrak{B} \mathfrak{c}_j^t)_{-1}, \quad 1 \leqslant j \leqslant r. \tag{3}$$

It is easily seen that the k-homomorphism $F^n \times F^n \to \operatorname{Tr} C$, $(\mathfrak{x}, \mathfrak{z}) \mapsto \overline{\left(\begin{smallmatrix} 0 & \mathfrak{x}x \\ 0 & \mathfrak{z} \end{smallmatrix}\right)}$, is injective. If we denote by φ_i^{12} resp. φ_i^{22} the map $F^n \to k$, $\mathfrak{x} \to \varphi_i(\overline{\left(\begin{smallmatrix} 0 & \mathfrak{x}x \\ 0 & 0 \end{smallmatrix}\right)})$ resp. $F^n \to k$, $\mathfrak{z} \mapsto \varphi_i(\overline{\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathfrak{z} \end{smallmatrix}\right)})$, then (3) is equivalent to the system

$$\sum_{i=1}^{r} \varphi_i^{12}(\mathfrak{x}_j \alpha_{ji}) = (\mathfrak{x}_j \mathfrak{a}_j^t)_{-1} \qquad \text{for all} \quad \mathfrak{x}_j \in F^n, \tag{I}$$

$$\sum_{i=1}^{r} \varphi_i^{12}(\mathfrak{y}_j \alpha_{ji} \mathfrak{B}) = (\mathfrak{y}_j \mathfrak{B} \mathfrak{a}_j^t)_{-1} \qquad \text{for all} \quad \mathfrak{y}_j \in F^n, \tag{I'}$$

$$\sum_{i=1}^{r} \varphi_i^{12}(\mathfrak{z}_j(\beta_{ji} \mathfrak{C} + \gamma_{ji} \mathfrak{B}) + \mathfrak{z}_j' \gamma_{ji}) + \sum_{i=1}^{r} \varphi_i^{22}(\mathfrak{z}_j \delta_{ji})$$
$$= (\mathfrak{z}_j (\mathfrak{b}_j^t + \mathfrak{B} \mathfrak{c}_j^t))_{-1} \qquad \text{for all} \quad \mathfrak{z}_j \in F^n. \tag{II}$$

Equation (I′) is a special case of (I), hence may be dropped. By Theorem 20 there exist vectors $\mathfrak{p}_i$, $\mathfrak{q}_i \in F^n$, $1 \leqslant i \leqslant r$, satisfying

$$\sum_{i=1}^{r} \alpha_{ji}\mathfrak{p}_i^t = \mathfrak{a}_j^t, \qquad 1 \leqslant j \leqslant r. \tag{$\tilde{\mathrm{I}}$}$$

$$\sum_{i=1}^{r} [(\beta_{ji}\mathfrak{E} + \gamma_{ji}\mathfrak{B})\mathfrak{p}_i^t - (\gamma_{ji}\mathfrak{p}_i^t)'] + \sum_{i=1}^{r} \delta_{ji}\mathfrak{q}_i^t$$
$$= \mathfrak{b}_j^t + \mathfrak{B}\mathfrak{c}_j^t, \qquad 1 \leqslant j \leqslant r. \tag{$\widetilde{\mathrm{II}}$}$$

Now a straightforward calculation shows that the elements

$$g_i := \overline{\begin{pmatrix} \mathfrak{p}_i^t & x\mathfrak{q}_i^t \\ 0 & 0 \end{pmatrix}} \in C, \qquad 1 \leqslant i \leqslant r,$$

solve the equations $\sum_{i=1}^{r} g_i a_{ji} = d_j$, $1 \leqslant j \leqslant r$.

By Theorem 1 there exists an AR-sequence $0 \to C \to B \to C \to 0$ in mod R. Because the k-dimension of $(\operatorname{Tr} C, k)$ is strictly larger than that of C, $\tilde{\sigma}$ is not an isomorphism, hence this sequence is not an AR-sequence in Mod R.

(B) Next let C_R be an indecomposable of dimension type (n, n), $n \geqslant 1$, in the uniserial subcategory $\mathfrak{u}$. This time we shall establish an isomorphism $C \to (\operatorname{Tr} C)^0$ and thereby show that the AR-sequence $0 \to C \cong (\operatorname{Tr} C)^0 \to B \to C \to 0$ in Mod R consists of finitely generated modules. Let

$$\mathfrak{C} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \in F^{n \times n}, \quad \mathfrak{E} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in F^{n \times n}$$

and $\mathfrak{A} = \mathfrak{E}x + \mathfrak{C}y \in R^{n \times n}$. It is easily seen that the exact sequence $0 \to e_2R^n \to^a e_1R^n \to^{v=\mathrm{can}} C \to 0$, where $a((e_2r_i)^t_{1 \leqslant i \leqslant n}) = \mathfrak{A} \cdot (e_2r_i)^t_{1 \leqslant i \leqslant n}$, is a minimal projective resolution of C. Hence $S := \operatorname{End}(C_R)$ may be identified with the set of all matrices $\mathfrak{S} \in F^{n \times n}$ such that there exists $\mathfrak{T} \in F^{n \times n}$ with $\mathfrak{S}\mathfrak{A} = \mathfrak{A}\mathfrak{T}$. Obviously the exact sequence $0 \to Re_1^n \to^{a^*} Re_2^n \to^w \operatorname{Tr} C \to 0$ with $((s_ie_1)_{1 \leqslant i \leqslant n})a^* := ((s_ie_1)_{1 \leqslant i \leqslant n}) \cdot \mathfrak{A}$ is a minimal projective resolution of $\operatorname{Tr} C$, and the endomorphism ring T of $\operatorname{Tr} C$ is the set of all $\mathfrak{T} \in F^{n \times n}$ for which there is $\mathfrak{S} \in F^{n \times n}$ with $\mathfrak{A}\mathfrak{T} = \mathfrak{S}\mathfrak{A}$. Straightforward computations show that $S = T = \{\mathfrak{X} \in F^{n \times n} \mid \mathfrak{C}\mathfrak{X}'\mathfrak{C} = \mathfrak{C}\mathfrak{X} - \mathfrak{X}\mathfrak{C}\} = \{(\xi_{ij}) \in F^{n \times n} \mid \xi_{ij} = 0$ for $i > j$ and $\xi_{ij} = \xi_{i-1,j-1} + \xi'_{i,j-1}$ for $1 \leqslant i \leqslant j \leqslant n\}$. It follows that a matrix $\mathfrak{T} = (\xi_{ij}) \in T$ is uniquely determined by its first row $(\xi_{11}, \ldots, \xi_{1n})$; to stress this fact, we write $\mathfrak{T}=\mathfrak{T}(\xi_{11}, ..., \xi_{1n})$. The kth power of the Jacobson radical of T is equal to $J(T)^k=\mathfrak{C}^k T=T\mathfrak{C}^k = \{\mathfrak{T}(\xi_1, ..., \xi_n) \mid (\xi_1, ..., \xi_n) \in F^n,\ \xi_1=\cdots=\xi_k=0\}$ and $J(T)^k/J(T)^{k+1}\cong F$, $0\leqslant k\leqslant n-1$. As a consequence, the local ring T is uniserial of length n, hence a minimal injective cogenerator on either side; in particular $(\mathrm{Tr}\ C)^0=(\mathrm{Tr}\ C_T, T_T)$.

In a second step we shall define a homomorphism $C\to(\mathrm{Tr}\ C)^0$. Because $C\otimes_R \mathrm{Tr}\ C\cong N^{n\times n}/U$, where $U=F^{n\times n}\mathfrak{A}+\mathfrak{A}F^{n\times n}$, we have to indicate a map ψ: $N^{n\times n}/U\to T$. First we consider the map φ: $F^{n\times n}\to T$, $\varphi(\mathfrak{X})=\sum_{k=1}^n \mathfrak{C}^{n-k}\cdot\mathfrak{T}(x_k)$, where x_k denotes the kth row of $\mathfrak{X}$. We shall show that φ satisfies two linearity properties and, as a consequence, that U is contained in its kernel. In virtue of the definition of T the map λ: $T\to T$, $\lambda(\mathfrak{T})=\mathfrak{T}+\mathfrak{C}\mathfrak{T}'$, defines an automorphism of T, the inverse of which is given by $\lambda^{-1}(\mathfrak{S})=\mathfrak{S}-\mathfrak{S}'\mathfrak{C}$. Note that $\mathfrak{C}\mathfrak{T}=\lambda(\mathfrak{T})\mathfrak{C}$ for all $\mathfrak{T}\in T$, hence $\mathfrak{C}^q\mathfrak{T}=\lambda^q(\mathfrak{T})\mathfrak{C}^q$ for all $q\geqslant 0$. Furthermore, in each equation $\mathfrak{A}\mathfrak{T}=\mathfrak{S}\mathfrak{A}$ with $\mathfrak{S}, \mathfrak{T}\in T$, we have $\mathfrak{S}=\lambda(\mathfrak{T})$ and $\mathfrak{T}=\lambda^{-1}(\mathfrak{S})$.

CLAIM 1. *φ is right T-linear and left λ^{n-2}-semilinear, i.e., $\varphi(\mathfrak{T}\mathfrak{X})=\lambda^{n-2}(\mathfrak{T})\,\varphi(\mathfrak{X})$ for $\mathfrak{T}\in T$ and $\mathfrak{X}\in F^{n\times n}$. (Of course we may assume $n\geqslant 2$.)*

Proof. Let $\mathfrak{S}\in T$. Again denoting by $x_1, ..., x_n$ the rows of $\mathfrak{X}$, the rows of $\mathfrak{X}\mathfrak{S}$ are $x_1\mathfrak{S}, ..., x_n\mathfrak{S}$ and since $\mathfrak{T}(x_k\mathfrak{S})=\mathfrak{T}(x_k)\mathfrak{S}$ we infer $\varphi(\mathfrak{X}\mathfrak{S})=\sum_{k=1}^n \mathfrak{C}^{n-k}\mathfrak{T}(x_k\mathfrak{S})=\sum_{k=1}^n \mathfrak{C}^{n-k}\mathfrak{T}(x_k)\mathfrak{S}=\varphi(\mathfrak{X})\mathfrak{S}$. Now we show the λ^{n-2}-semilinearity on the left side. Taking into account that $\varphi(\mathfrak{C}^i\mathfrak{X})=\mathfrak{C}^i\varphi(\mathfrak{X}) = \lambda^{n-2}(\mathfrak{C}^i)\,\varphi(\mathfrak{X})$ for $1\leqslant i\leqslant n$ and that each $\mathfrak{T}=\mathfrak{T}(\sigma_1, ..., \sigma_n)\in T$ may be written in the form $\mathfrak{T}=\sum_{i=1}^n \mathfrak{T}(\sigma_i, 0, ..., 0)\mathfrak{C}^{i-1}$, we have only to show that $\varphi(\mathfrak{T}(\sigma, 0, ..., 0)\mathfrak{X})=\lambda^{n-2}(\mathfrak{T}(\sigma, 0, ..., 0))\,\varphi(\mathfrak{X})$ for all $\sigma\in F$. We derive a formula for $\varphi(\mathfrak{S}\mathfrak{X})$, where $\mathfrak{S}=(\sigma_{ij})\in F^{n\times n}$ is some matrix. The rows of $\mathfrak{S}\mathfrak{X}$ are $\sum_{j=1}^n \sigma_{kj}x_j$, $1\leqslant k\leqslant n$, hence

$$\begin{aligned}
\varphi(\mathfrak{S}\mathfrak{X}) &= \sum_{k=1}^n \mathfrak{C}^{n-k}\mathfrak{T}\left(\sum_{j=1}^n \sigma_{kj}x_j\right) \\
&= \sum_{1\leqslant j,k\leqslant n} \mathfrak{C}^{n-k}\mathfrak{T}(\sigma_{kj}x_j) \\
&= \sum_{1\leqslant j,k\leqslant n} \mathfrak{C}^{n-k}\mathfrak{T}(\sigma_{kj}, 0, ..., 0)\,\mathfrak{T}(x_j) \\
&= \sum_{j=1}^n \left(\sum_{k=1}^n \mathfrak{C}^{n-k}\mathfrak{T}(\sigma_{kj}, 0, ..., 0)\right)\mathfrak{T}(x_j) \\
&= \sum_{j=1}^n \varphi(\mathfrak{S}_j)\cdot\mathfrak{T}(x_j),
\end{aligned}$$

where

$$\mathfrak{S}_j = \begin{pmatrix} \sigma_{1j} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \sigma_{nj} & 0 & \cdots & 0 \end{pmatrix} \in F^{n\times n}.$$

Now let us deal with the specific case $\mathfrak{S} = \mathfrak{T}(\sigma, 0, ..., 0)$. Let s_j denote the jth column of $\mathfrak{S}$ and $\mathfrak{S}_j = (s_j, 0, ..., 0) \in F^{n\times n}$. Then $s_1 = (\sigma, 0, ..., 0)'$ and $s_j = (0, \sigma^{(j-2)}, \binom{j-2}{1}\sigma^{(j-3)}, ..., \binom{j-2}{j-3}\sigma', \sigma, 0, ..., 0)'$, $2 \leqslant j \leqslant n$, hence $\varphi(\mathfrak{S}_1) = \mathfrak{C}^{n-1}\mathfrak{T}(\sigma, 0, ..., 0) = \lambda^{n-1}(\mathfrak{T}(\sigma, 0, ..., 0))\mathfrak{C}^{n-1} = \lambda^{n-2}(\mathfrak{T}(\sigma, 0, ..., 0) + \mathfrak{C}\mathfrak{T}(\sigma', 0, ..., 0))\mathfrak{C}^{n-1} = \lambda^{n-2}(\mathfrak{T}(\sigma, 0, ..., 0))\,\mathfrak{C}^{n-1}$, and $\varphi(\mathfrak{S}_j) = \sum_{k=2}^{j} \mathfrak{C}^{n-k}\binom{j-2}{k-2}\,\mathfrak{T}(\sigma^{(j-k)}, 0, ..., 0) = \mathfrak{C}^{n-j}\sum_{k=2}^{j} \mathfrak{C}^{j-k}\binom{j-2}{k-2}\,\mathfrak{T}(\sigma^{(j-k)}, 0, ..., 0) = \mathfrak{C}^{n-j}\lambda^{j-2}(\mathfrak{T}(\sigma, 0, ..., 0) = \lambda^{n-j}(\lambda^{j-2}(\mathfrak{T}(\sigma, 0, ..., 0)))\mathfrak{C}^{n-j} = \lambda^{n-2}(\mathfrak{T}(\sigma, 0, ..., 0))\,\mathfrak{C}^{n-j}$ for $2 \leqslant j \leqslant n$. Consequently $\varphi(\mathfrak{S}\mathfrak{X}) = \sum_{j=1}^{n} \varphi(\mathfrak{S}_j)\,\mathfrak{T}(x_j) = \lambda^{n-2}(\mathfrak{T}(\sigma, 0, ..., 0))\,\mathfrak{C}^{n-1}\mathfrak{T}(x_1) + \sum_{j=2}^{n} \lambda^{n-2}(\mathfrak{T}(\sigma, 0, ..., 0))\,\mathfrak{C}^{n-j}\mathfrak{T}(x_j) = \lambda^{n-2}(\mathfrak{T}(\sigma, 0, ..., 0))\cdot\varphi(\mathfrak{X})$.

With the help of φ we define $\psi\colon N^{n\times n} \to T$ by $\psi(x\mathfrak{X} + y\mathfrak{Y}) = \varphi(\mathfrak{Y} - \mathfrak{C}\mathfrak{X})$. It is easily checked that ψ is T-linear on the right and λ^{n-1}-semilinear on the left side, i.e., $\psi(\mathfrak{T}(x\mathfrak{X} + y\mathfrak{Y})) = \lambda^{n-1}(\mathfrak{T})\,\psi(x\mathfrak{X} + y\mathfrak{Y})$ for all $\mathfrak{T} \in T$. Because $\psi(\mathfrak{X}\mathfrak{A}) = \psi(x(\mathfrak{X} - \mathfrak{X}'\mathfrak{C}) + y\mathfrak{X}\mathfrak{C}) = \varphi(\mathfrak{X}\mathfrak{C} - \mathfrak{C}(\mathfrak{X} - \mathfrak{X}'\mathfrak{C})) = \varphi(\mathfrak{X})\mathfrak{C} - \mathfrak{C}\varphi(\mathfrak{X}) - \mathfrak{C}\varphi(\mathfrak{X})'\,\mathfrak{C} = 0$ and $\psi(\mathfrak{A}\mathfrak{X}) = \psi(x\mathfrak{X} + y\mathfrak{C}\mathfrak{X}) = \varphi(\mathfrak{C}\mathfrak{X} - \mathfrak{C}\mathfrak{X}) = 0$ for all $\mathfrak{X} \in F^{n\times n}$, ψ induces a unique map $N^{n\times n}/U \to T$ which we denote by ψ, also. By composition we arrive at the map $\sigma\colon C \times \operatorname{Tr} C \to N^{n\times n}/U \to^{\psi} T$ which is explicitly given by

$$\sigma\left(\overline{\begin{pmatrix} \mathfrak{a}' & x\mathfrak{b}' + y\mathfrak{c}' \\ 0 & 0 \end{pmatrix}}, \overline{\begin{pmatrix} 0 & \mathfrak{x}x + \mathfrak{y}y \\ 0 & \mathfrak{z} \end{pmatrix}}\right)$$
$$= \varphi((\mathfrak{a}'\mathfrak{y} + \mathfrak{c}'\mathfrak{z}) - \mathfrak{C}(\mathfrak{a}'\mathfrak{x} + \mathfrak{b}'\mathfrak{z} - (\mathfrak{a}'\mathfrak{y})'))$$

and which is T-linear on the right and λ^{n-1}-semilinear on the left side. The proof of the next assertion will complete part B.

CLAIM 2. *The map* $\tilde{\sigma}\colon C \to (\operatorname{Tr} C)^0$ *induced by* σ *is an isomorphism.*

Proof. To show that $\tilde{\sigma}$ is a monomorphism, we note first that for all $\mathfrak{x} \in F^n$ the equation $\varphi((\mathfrak{x}', 0, ..., 0)) = 0$ implies $\mathfrak{x} = 0$. Now let

$$c = \overline{\begin{pmatrix} \mathfrak{a}' & x\mathfrak{b}' + y\mathfrak{c}' \\ 0 & 0 \end{pmatrix}} \in C$$

with $\tilde{\sigma}(c) = 0$. This means that for all $d = \overline{\left(\begin{smallmatrix} 0 & \mathfrak{x}x + \mathfrak{y}y \\ 0 & \mathfrak{z} \end{smallmatrix}\right)} \in \operatorname{Tr} C$ we have $\varphi((\mathfrak{a}'\mathfrak{y} + \mathfrak{c}'\mathfrak{z}) - \mathfrak{C}(\mathfrak{a}'\mathfrak{x} + \mathfrak{b}'\mathfrak{z} - (\mathfrak{a}'\mathfrak{y})')) = 0$. Putting $\mathfrak{x} = 0$, $\mathfrak{z} = 0$, $\mathfrak{y} = (1, 0, ..., 0)$ we obtain $\varphi((\mathfrak{a}' + \mathfrak{C}\mathfrak{a}'')\mathfrak{y}) = 0$, hence $\mathfrak{a}' + \mathfrak{C}\mathfrak{a}'' = 0$ and $\mathfrak{a}' = 0$. Similarly, we get $\mathfrak{c}' - \mathfrak{C}\mathfrak{b}' = 0$, hence

$$c=\overline{\begin{pmatrix}0 & x\mathfrak{b}'+y\mathfrak{C}\mathfrak{b}'\\ 0 & 0\end{pmatrix}}=\overline{\mathfrak{A}e_1\mathfrak{b}'}=0.$$

We shall demonstrate surjectivity of $\tilde{\sigma}$ by showing that $\tilde{\sigma}$ maps a T-basis of C onto a T-basis of $(\operatorname{Tr} C)^0$. It is easy to see that the elements $c_1 = v((0, ..., 0, e_1)')$ and $c_2 = v((0, ..., 0, y)')$ from a basis of C, viewed as a left T-module, and the elements $d_1 = w((e_2, 0, ..., 0))$ and $d_2 = w((y, 0, ..., 0))$ a basis of $\operatorname{Tr} C$, viewed as a right T-module. The equations $\tilde{\sigma}(c_1)(d_1) = 0$, $\tilde{\sigma}(c_1)(d_2) = \mathfrak{C}$, $\tilde{\sigma}(c_2)(d_1) = \mathfrak{C}$, and $\tilde{\sigma}(c_2)(d_2) = 0$ show that the basis $(\tilde{\sigma}(c_1), \tilde{\sigma}(c_2))$ is the dual of the basis (d_2, d_1), i.e., $\tilde{\sigma}$ maps the basis (c_1, c_2) onto the basis $(\tilde{\sigma}(c_1), \tilde{\sigma}(c_2))$. Because $\tilde{\sigma}$, viewed as a map between left T-modules is in addition λ^{n-1}-semilinear, we may conclude that $\tilde{\sigma}$ is an isomorphism.

(C) Finally we shall deal with the indecomposables of dimension type (m, n), where $|m-n| = 1$. We confine ourselves to modules of dimension-type $(n, n-1)$, $n \geqslant 1$; the case $(n, n+1)$ is handled similarly. First we note that $I_1 = e_1 R/e_1 J$ and $I_2 = D(Re_2) = (Re_{2F}, F_F)$ up to isomorphism are all indecomposable injective right R-modules. By Theorem 4 there exists an AR-sequence $0 \to A \to B \to I_1 \to 0$ in $\operatorname{mod} R$ with $A \cong D\operatorname{Tr}((DI_1)^0)$. Because D is weakly symmetric, we have $DI_1 = Re_1$, hence $(DI_1)^0 \cong I_1$ and $A \cong D\operatorname{Tr} I_1$. To find an AR-sequence in $\operatorname{mod} R$, ending with I_2, we note that there exists an AR-sequence $0 \to Re_2 \to E \to \operatorname{Tr} I_2 \to 0$ in $R\operatorname{mod}$ [10, Folgerung 9]; the dual $0 \to D\operatorname{Tr} I_2 \to DE \to I_2 \to 0$ is an AR-sequence in $\operatorname{mod} R$. Now let C_R be an indecomposable of dimension-type $(n, n-1)$, $n \geqslant 1$. Because the dimension type of I_1 resp. I_2 is $(1, 0)$ resp. $(2, 1)$ and the Coxeter transformation belonging to the Coxeter functor $C^+ \cong D\operatorname{Tr}$ is $\left(\begin{smallmatrix}3 & -2\\ 2 & -1\end{smallmatrix}\right)$, we have $C \cong (D\operatorname{Tr})^m I_1$ and an AR-sequence $0 \to (D\operatorname{Tr})^{m+1} I_1 \to U \to (D\operatorname{Tr})^m I_1 \cong C \to 0$ in $\operatorname{mod} R$ if $n = 2m+1$, whereas in case $n = 2m$ we have $C \cong (D\operatorname{Tr})^{m-1} I_2$ and an AR-sequence $0 \to (D\operatorname{Tr})^m I_2 \to V \to (D\operatorname{Tr})^{m-1} I_2 \cong C \to 0$ in $\operatorname{mod} R$. It remains to prove that these sequences are even AR-sequences in $\operatorname{Mod} R$, i.e., that $(\operatorname{Tr} C)^0$ is finitely generated. To this end let

$$\mathfrak{P}=\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0\,1 & 0\end{pmatrix}\in F^{n\times(n+1)},$$

$$\mathfrak{Q}=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ & & \ddots & \ddots & 0\\ 0 & \cdots & & 0 & 1\end{pmatrix}\in F^{n\times(n+1)},$$

and $\mathfrak{A} = \mathfrak{P}x + \mathfrak{Q}y \in R^{n\times(n+1)}$. The exact sequence $0 \to e_2R^{n+1} \to^a e_1R^n \to^w C \to 0$, where $a((e_2r_i)^t_{1\leq i\leq n+1}) = \mathfrak{A}\cdot(e_2r_i)^t_{1\leq i\leq n+1}$, is a minimal projective resolution of C. Dualization with respect to R yields the exact sequence $0 \to Re_1^n \to^{a^*} Re_2^{n+1} \to \operatorname{Tr} C \to 0$, hence the endomorphism ring T of $\operatorname{Tr} C$ may be identified with the set of matrices $\mathfrak{X} \in F^{(n+1)\times(n+1)}$ such that there is $\mathfrak{Y} \in F^{n\times n}$ with $\mathfrak{A}\mathfrak{X} = \mathfrak{Y}\mathfrak{A}$, i.e., $\mathfrak{P}\mathfrak{X} = \mathfrak{Y}\mathfrak{P} - \mathfrak{Y}'\mathfrak{Q}$ and $\mathfrak{Q}\mathfrak{X} = \mathfrak{Y}\mathfrak{Q}$. Analyzing these equations and using the notations of (B) we see that $\mathfrak{X} = \mathfrak{T}(\alpha, -\binom{n}{1}\alpha', \binom{n}{2}\alpha'', ..., (-1)^n\alpha^{(n)})$ and $\mathfrak{Y} = \mathfrak{T}(\alpha, -\binom{n-1}{1}\alpha', \binom{n-1}{2}\alpha'', ..., (-1)^{n-1}\alpha^{(n-1)})$ for some $\alpha \in F$. In particular, F is isomorphic to T via $\alpha \mapsto \mathfrak{T}(\alpha, -\binom{n}{1}\alpha', ..., (-1)^n\alpha^{(n)})$ and $(\operatorname{Tr} C)^0 = (\operatorname{Tr} C_T, T_T)$. Because w is a T-epimorphism, it induces an embedding $(\operatorname{Tr} C)^0 \to (Re_{2T}^{n+1}, T_T)$, hence it is sufficient to show that (Re_{2T}^{n+1}, T_T) is finitely generated. This is easily done by induction.

Corollary 14. *For each finitely generated indecomposable, non-injective right R-module A there exists an Auslander–Reiten sequence $0 \to A \to B \to C \to 0$ in* mod R.

Proof. Bearing in mind Ringel's classification of the finitely generated indecomposable right R-modules, this is an immediate consequence of the preceding theorem.

3

The last section is devoted to the proofs of two results concerning systems of differential equations over the field $k((X))$, which are crucial for Theorem 13. As in Section 2, $F = k((X))$ denotes the field of formal Laurent series in the variable X over a field k of characteristic 0 and $D\xi = \xi'$ the usual derivative of some $\xi \in F$. Recursively we define $\xi^{(0)} = \xi$ and $\xi^{(i+1)} := \xi^{(i)\prime}$ for $i \geq 0$. Obviously $\xi' = 0$ if and only if $\xi \in k$.

Theorem 15. *For all $n \geq 1$ and $\mathfrak{C} \in F^{n\times n}$ the k-vector space of all solutions $\mathfrak{x} \in F^n$ of the equation $\mathfrak{x}'^t = \mathfrak{C}\mathfrak{x}^t$ has finite k-dimension.*

In the proof of this theorem and in the remaining part of this section differential operators will play a fundamental role. Letting $\alpha_0, ..., \alpha_l \in F$ we call the k-linear map Δ: $F \to F$ given by $\Delta(\xi) = \sum_{i=0}^{l} \alpha_i \cdot \xi^{(i)}$ a differential operator over F with coefficients $\alpha_0, ..., \alpha_l$. Obviously $\Delta = 0$ if and only if $\alpha_0 = \cdots = \alpha_l = 0$. α_0 is called the constant coefficient; if $\alpha_l \neq 0$, then $\deg \Delta = l$ is called the degree of Δ.

Lemma 16. *Let $\Delta \neq 0$ be a differential operator over F with coefficients $\alpha_0, ..., \alpha_l \in F$. Then the kernel of Δ is finite-dimensional over k.*

Proof. Let $\xi=\sum_{m\geqslant e}\xi_m X^m\in F$ with $e\in\mathbb{Z}$ and $\xi_m\in k$ and $\Delta(\xi)=\sum_{t\in\mathbb{Z}}\Delta(\xi)_t X^t$ with $\Delta(\xi)_t\in k$. First we shall derive a representation of $\Delta(\xi)_t$ as a linear map of the coefficients $(\xi_m)_{m\geqslant e}$ of ξ. Let $\alpha_i=\sum_{j\geqslant a_i}\alpha_{ij}X^j$ with $a_i\in\mathbb{Z}$ and $\alpha_{ij}\in k$, and let $\alpha_{ia_i}\neq 0$ if $\alpha_i\neq 0$. For some $m\in\mathbb{Z}$ and $i\geqslant 0$ we put

$$(m)_i=\begin{cases}1 & \text{if } i=0\\ (m+1)\cdot\cdots\cdot(m+i) & \text{if } i\geqslant 1\end{cases}.$$

Then $\xi^{(i)}=\sum_{m\geqslant e-i}(m)_i\,\xi_{m+i}X^m$, hence

$$\begin{aligned}\Delta(\xi)_t&=\sum_{i=0}^{l}\sum_{\substack{j\geqslant a_i\\ m\geqslant e-i\\ j+m=t}}\alpha_{ij}(m)_i\,\xi_{m+i}\\ &=\sum_{i=0}^{l}\sum_{t-e+i\geqslant j\geqslant a_i}\alpha_{ij}(t-j)_i\,\xi_{t-j+i}.\end{aligned}$$

We may choose an index $0\leqslant i_\Delta\leqslant l$ such that $\delta:=\delta_\Delta:=a_{i_\Delta}-i_\Delta$ is minimal among the numbers a_i-i with $\alpha_i\neq 0$, and we denote by I the set of all $0\leqslant i\leqslant l$ with $\alpha_i\neq 0$ and $a_i-i=\delta$. Then $w(t):=w_\Delta(t):=\sum_{i\in I}\alpha_{ia_i}(t-a_i)_i$ is a polynomial $\neq 0$ in t with coefficients in k of degree $\leqslant l$, and there exists a k-linear map $h_t(\xi_e,\ldots,\xi_{t-\delta-1})$ with coefficients in k such that

$$\Delta(\xi)_t=w(t)\,\xi_{t-\delta}+h_t(\xi_e,\ldots,\xi_{t-\delta-1})\tag{4}$$

for $t\geqslant e+\delta$.

Now the assertion of the lemma is quickly shown. We may assume that there is an element $\xi=\sum_{m\geqslant e}\xi_m X^m\in\operatorname{Ker}\Delta$ with $\xi_e\neq 0$. Then we may infer from the equation $0=\Delta(\xi)_{e+\delta}=w(e+\delta)\xi_e$ that $e+\delta$ is a root of w. On the other hand, if v denotes the largest root of w, then formula (4) shows that the $\xi_t\in k$, $t>v-\delta$, are uniquely determined by $\xi_e,\ldots,\xi_{v-\delta}$. Because w has only a finite number of roots, these two remarks finish our proof.

Similar arguments settle the next lemma, which is recorded for later use.

LEMMA 17. *For a differential operator $\Delta\neq 0$ the following statements are equivalent.*

(1) $\operatorname{Ker}\Delta=0$.

(2) *For each $\eta\in F$ the equation $\Delta(\xi)=\eta$ has a unique solution.*

(3) *w has no roots in $\mathbb{Z}$.*

If one of the conditions holds, we shall call Δ regular.

Proof. (1) $\Rightarrow$ (3) If we assume that w has a root in $\mathbb{Z}$, there is a largest one, say v. Let $e=v-\delta$ and $\xi_e=1$. Then, using (4) we can recursively calculate a unique family $(\xi_m)_{m \geqslant e}$ in k such that $\xi = \sum_{m \leqslant e} \xi_m X^m \in F$ is an element $\neq 0$ of Ker Δ.

(3) $\Rightarrow$ (2) Given $\eta \in F$, Eq. (4) allows us to compute the coefficients $(\xi_m)_{m \geqslant e}$ of a unique $\xi = \sum_{m \geqslant e} \xi_m X^m$ with $\Delta(\xi) = \eta$.

(2) $\Rightarrow$ (1) Obvious.

Proof of Theorem 15. We have to show that the k-vector space $\mathfrak{R} = \{\mathfrak{x} \in F^n \mid \mathfrak{x}'' = \mathfrak{C}\mathfrak{x}'\}$ has finite dimension. We define a sequence $(\mathfrak{C}_i)_{i \geqslant 1}$ of matrices in $F^{n \times n}$ as follows: $\mathfrak{C}_1 = \mathfrak{C}$ and $\mathfrak{C}_{i+1} = \mathfrak{C}_i' + \mathfrak{C}_i \mathfrak{C}$ for $i \geqslant 1$. Note that $\mathfrak{x}^{(i)\prime} = \mathfrak{C}_i \mathfrak{x}'$ for all $\mathfrak{x} \in \mathfrak{R}$. Because $\mathfrak{C}_1, \ldots, \mathfrak{C}_{n^2+1}$ are linearly dependent over F, there exist $\alpha_1, \ldots, \alpha_{n^2+1} \in F$, some of which are $\neq 0$, such that $\sum_{i=1}^{n^2+1} \alpha_i \mathfrak{C}_i = 0$. For all $\mathfrak{x} \in \mathfrak{R}$ we infer $\sum_{i=1}^{n^2+1} \alpha_i \mathfrak{x}^{(i)\prime} = \sum_{i=1}^{n^2+1} \alpha_i \mathfrak{C}_i \mathfrak{x}' = 0$. Hence $\mathfrak{R} \subset (\operatorname{Ker} \Delta)^n$, where Δ denotes the differential operator over F, which is given by $\Delta(\xi) = \sum_{i=1}^{n^2+1} \alpha_i \xi^{(i)}$. Because $\Delta \neq 0$, Ker Δ has finite k-dimension according to Lemma 16, hence $\mathfrak{R}$ has finite k-dimension, also.

The following result by McConnell and Robson [7] is an obvious consequence of Theorem 15. As usual, $k(X)$ denotes the field of rational functions in the variable X over k.

COROLLARY 18 [7, Theorem 2.3]. *For each $\mathfrak{C} \in k(X)^{n \times n}$ the k-vector space of all $\mathfrak{x} \in F^n$, solving the equation $\mathfrak{x}'' = \mathfrak{C}\mathfrak{x}'$, is finite-dimensional.*

The next theorem, the proof of which is rather intricate, deals with the solvability of certain systems of non-homogeneous differential equations which arose in the proof of Theorem 13. An essential ingredient in its formulation and its proof is the notion of the adjoint of a differential operator. Let $\Delta(\xi) = \sum_{i=0}^{l} \alpha_i \xi^{(i)}$ be a differential operator over F. Then $\tilde{\Delta}(\xi) = \sum_{i=0}^{l} (-1)^i (\alpha_i \xi)^{(i)}$ defines a differential operator, which we call the adjoint of Δ. For instance $\tilde{D} = -D$. If $\Delta \neq 0$, then $\tilde{\Delta} \neq 0$ and the degrees of Δ and $\tilde{\Delta}$ coincide. To justify the adjective "adjoint" we consider the k-bilinear map $F \times F \to k$, $(\xi, \eta) \to (\xi\eta)_{-1}$. (For some $\alpha \in F$ we denote by $\alpha_m \in k$, $m \in \mathbb{Z}$, the coefficient at X^m.) Because $\xi_m = (\xi X^{-(m+1)})_{-1}$ for all $m \in \mathbb{Z}$, this map is not degenerated. Noting that $(\alpha')_{-1} = 0$ for all $\alpha \in F$ and $(\xi\eta)' = \xi'\eta + \xi\eta'$, we find $(\xi'\eta)_{-1} = -(\xi\eta')_{-1}$, inductively $(\xi^{(i)}\eta)_{-1} = (-1)^i (\xi\eta^{(i)})_{-1}$ for all $i \geqslant 1$, hence $(\Delta(\xi) \cdot \eta)_{-1} = \sum_{i=0}^{l} (\alpha_i \xi^{(i)} \eta)_{-1} = \sum_{i=1}^{l} (-1)^i (\xi(\alpha_i \eta)^{(i)})_{-1} = (\xi \cdot \tilde{\Delta}(\eta))_{-1}$. As a first consequence of this equality, we conclude that $\tilde{\tilde{\Delta}} = \Delta$. For later use we show that $\tilde{\Delta}$ is regular if and only if Δ has this property. Let Δ be regular and $\eta \in F$ such that $\tilde{\Delta}(\eta) = 0$. Since for each $m \in \mathbb{Z}$ there exists $\lambda_m \in F$ with $\Delta(\lambda_m) = X^m$, we have $\eta_m = (\eta X^{-(m+1)})_{-1} = (\eta\, \Delta(\lambda_{-(m+1)}))_{-1} = (\tilde{\Delta}(\eta)\, \lambda_{-(m+1)})_{-1} = 0$ for all $m \in \mathbb{Z}$, hence $\eta = 0$. Because $\tilde{\tilde{\Delta}} = \Delta$, the converse implication is obvious.

THEOREM 19. *Let $(\Delta_{ji})_{1 \leqslant j, i \leqslant n}$ be a family of differential operators over F and $\beta_1, \ldots, \beta_n \in F$. There exist $\sigma_1, \ldots, \sigma_n \in F$ satisfying the equations*

$\sum_{i=1}^{n} \tilde{\Delta}_{ji}(\sigma_i) = \beta_j$, $1 \leqslant j \leqslant n$, *if and only if there exist* $\varphi_1, ..., \varphi_n \in (F, k)$ *such that* $\sum_{i=1}^{n} \varphi_i(\Delta_{ji}(\xi_j)) = (\beta_j \xi_j)_{-1}$, $1 \leqslant j \leqslant n$, *for all* $\xi_1, ..., \xi_n \in F$.

One implication is trivial. Suppose that $\sigma_1, ..., \sigma_n$ solve the equations $\sum_{i=1}^{n} \tilde{\Delta}_{ji}(\sigma_i) = \beta_j$, $1 \leqslant j \leqslant n$. Then the linear forms $\varphi_i \in (F, k)$, given by $\varphi_i(\xi) = (\sigma_i \xi)_{-1}$, satisfy the equations $\sum_{i=1}^{n} \varphi_i(\Delta_{ji}(\xi_j)) = (\beta_j \xi_j)_{-1}$, $1 \leqslant j \leqslant n$.

The converse direction is settled by a type of elimination procedure. To describe it, first we have to introduce a number of technical manipulations with differential operators.

(1) The composition $\Delta_1 \Delta_2 = \Delta_1 \circ \Delta_2$ of differential operators Δ_1, Δ_2 over F is a differential operator, also. Obviously, if $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$, then $\Delta_1 \Delta_2 \neq 0$ and $\deg \Delta_1 \Delta_2 = \deg \Delta_1 + \deg \Delta_2$. Furthermore, it is easily seen that $\widetilde{\Delta_1 \Delta_2} = \tilde{\Delta}_2 \tilde{\Delta}_1$.

The next two constructions are special instances of (1). We assume that $\Delta(\xi) = \sum_{i=0}^{l} \alpha_i \xi^{(i)}$ is a differential operator over F.

(2) Let $\alpha \in F$. We define the differential operators $\alpha\Delta$ and $\Delta\alpha$ by $(\alpha\Delta)(\xi) = \alpha \cdot \Delta(\xi)$ and $(\Delta\alpha)(\xi) = \Delta(\alpha\xi)$, respectively. Since $(\alpha\xi)^{(i)} = \sum_{j=0}^{i} \binom{i}{j} \alpha^{(i-j)} \xi^{(j)}$, we have $(\Delta\alpha)(\xi) = \sum_{j=0}^{l} \Delta^{(j)}(\alpha) \xi^{(j)}$, where $\Delta^{(j)}(\alpha) = \sum_{j \leqslant i \leqslant l} \alpha_i \binom{i}{j} \alpha^{(i-j)}$. In particular $\Delta^{(0)}(\alpha) = \Delta(\alpha)$ and $\Delta^{(l)}(\alpha) = \alpha_l \cdot \alpha$. Hence, if $\Delta \neq 0$ and $\alpha \neq 0$, the degrees of Δ, $\alpha\Delta$, and $\Delta\alpha$ coincide. According to (1) we have $\widetilde{\alpha\Delta} = \tilde{\Delta}\alpha$ and $\widetilde{\Delta\alpha} = \alpha\tilde{\Delta}$.

(3) The operators $D\Delta$ and ΔD are defined by $D\Delta(\xi) = \Delta(\xi)' = \alpha_0' \xi + (\alpha_0 + \alpha_1')\xi' + \cdots + (\alpha_{l-1} + \alpha_l')\xi^{(l)} + \alpha_l \xi^{(l+1)}$ and $\Delta D(\xi) = \Delta(\xi')$, respectively. Because $\tilde{D} = -D$, the adjoints of $D\Delta$ and ΔD are $\widetilde{D\Delta} = -\tilde{\Delta} D$ and $\widetilde{\Delta D} = -D\tilde{\Delta}$.

(4) Now we assume that $\alpha_0 = 0$. We define Δ^b by $\Delta^b(\xi) = \sum_{i=1}^{l} \alpha_i \xi^{(i-1)}$. Since $\Delta^b D = \Delta$, we may infer $D\widetilde{\Delta^b} = -\widetilde{\Delta^b D} = -\tilde{\Delta}$.

For later use we fix the following special case of (4). If the constant coefficient of Δ lies in k, then the constant coefficient of $D\Delta$ is 0, hence $(D\Delta)^b D = D\Delta$.

A decisive technical tool in our proof of Theorem 19 is the k-linear map $s \colon F \to F$ given by $s(\sum_{m \geqslant e} \xi_m X^m) = \sum_{m \geqslant e, m \neq -1} (1/(m+1)) \xi_m X^{m+1} + \xi_{-1}$; it may be considered as a kind of a definite integral. It is easily checked that s is an isomorphism, the inverse of which is given by $s^{-1}(\xi) = \xi' + \xi_0 \cdot X^{-1}$; furthermore, $s(\xi)' = \xi - \xi_{-1} \cdot X^{-1}$ and $s(\xi') = \xi - \xi_0$. Hence, for some differential operator Δ over F we have the formula $s^{-1} \Delta(\xi) = \Delta(\xi)' + \Delta(\xi)_0 \cdot X^{-1} = D\Delta(\xi) + (\Delta(\xi) \cdot X^{-1})_{-1} \cdot X^{-1} = D\Delta(\xi) + (\tilde{\Delta}(X^{-1}) \cdot \xi)_{-1} \cdot X^{-1}$.

Now we may tackle the proof of the non-trivial implication of Theorem 19. We suppose that there are $\varphi_1, ..., \varphi_n \in (F, k)$ such that the system of equations

$$\sum_{i=1}^{n} \varphi_i(\Delta_{ji}(\xi_j)) = (\beta_j \xi_j)_{-1}, \qquad 1 \leqslant j \leqslant n, \tag{S}$$

holds for all $\xi_1, ..., \xi_n \in F$ and we have to show the existence of elements $\sigma_1, ..., \sigma_n \in F$ with $\sum_{i=1}^n \tilde{\Delta}_{ji}(\sigma_i) = \beta_j$, $1 \leqslant j \leqslant n$. In case such an n-tuple $(\sigma_1, ..., \sigma_n)$ exists, we call (S) solvable and $(\sigma_1, ..., \sigma_n)$ a solution.

It is our plan to transform (S) into a similar system, the solvability of which is guaranteed by some induction hypothesis and implies solvability of (S). The transformed system is attained by a sequence of "elementary operations" which we shall explain now. For short, we shall call a transformed system (S′) admissible, if solvability of (S) may be derived from solvability of (S′).

(A) (S′) results from (S) by a permutation of the indices $1 \leqslant j \leqslant n$ or $1 \leqslant i \leqslant n$.

(B) (S′) results from (S) by addition of rows in (S).

(C) Let $1 \leqslant q \leqslant n$, $0 \neq \alpha \in F$ and (S′) the system $(\varphi_q \alpha^{-1})(\alpha \Delta_{jq}(\xi_j)) + \sum_{i \neq q} \varphi_i(\Delta_{ji}(\xi_j)) = (\beta_j \xi_j)_{-1}$, $1 \leqslant j \leqslant n$. Because $\widetilde{\alpha \Delta_{jq}} = \tilde{\Delta}_{jq} \alpha$, $(\sigma_1, ..., \sigma_n)$ is a solution of (S′) iff $(\sigma_1, ..., \sigma_{q-1}, \alpha\sigma_q, \sigma_{q+1}, ..., \sigma_n)$ solves (S).

(D) Let $1 \leqslant p \leqslant n$, $0 \neq \alpha \in F$ and (S′) the system

$$\sum_{i=1}^n \varphi_i((\Delta_{pi}\alpha)(\xi_p)) = (\beta_p \alpha \xi_p)_{-1},$$

$$\sum_{i=1}^n \varphi_i(\Delta_{ji}(\xi_j)) = (\beta_j \xi_j)_{-1}, \qquad j \neq p.$$

Since $\widetilde{\Delta_{pi}\alpha} = \alpha \tilde{\Delta}_{pi}$, (S′) and (S) have the same solutions.

(E) We suppose that $\Delta_{pq} \neq 0$ for some $1 \leqslant p, q \leqslant n$ and that the constant coefficient of Δ_{pq} is 0. We may assume that $(p, q) = (1, 1)$ and that there exists $1 \leqslant l \leqslant n$ such that the constant coefficients of $\Delta_{11}, ..., \Delta_{1l}$ are 0, whereas those of $\Delta_{1l+1}, ..., \Delta_{1n}$ are $\neq 0$. We may even assume that the constant coefficients of $\Delta_{1l+1}, ..., \Delta_{1n}$ are equal to 1. The jth row of (S) can be written in the form $\sum_{i=1}^l \varphi_i(\Delta_{ji}(\xi_j)) + \sum_{i=l+1}^n \varphi_i s(s^{-1} \Delta_{ji}(\xi_j)) = (\beta_j \xi_j)_{-1}$. Because $s^{-1} \Delta_{ji}(\xi_j) = D\Delta_{ji}(\xi_j) + (\tilde{\Delta}_{ji}(X^{-1})\xi_j)_{-1} \cdot X^{-1}$, the latter equation transforms into $\sum_{i=1}^l \varphi_i(\Delta_{ji}(\xi_j)) + \sum_{i=l+1}^n \varphi_i s(D\Delta_{ji}(\xi_j)) = (\gamma_j \xi_j)_{-1}$, where $\gamma_j = \beta_j - \sum_{i=l+1}^n \varphi_i(1) \cdot \tilde{\Delta}_{ji}(X^{-1})$. Having in mind the assumptions on $\Delta_{11}, ..., \Delta_{1n}$ we may write $\Delta_{1i} = \Delta_{1i}^b D$ for $1 \leqslant i \leqslant l$ and $D\Delta_{1i} = (D\Delta_{1i})^b D$ for $1 + l \leqslant i \leqslant n$. As a consequence $\sum_{i=1}^l \varphi_i(\Delta_{1i}^b(\xi_1')) + \sum_{i=l+1}^n \varphi_i s(D\Delta_{1i})^b (\xi_1'))$ $= (\gamma_1 \xi_1)_{-1}$ for all $\xi_1 \in F$. The special choice $\xi_1 = 1$ yields $(\gamma_1)_{-1} = 0$, hence $\gamma_1 = s(\gamma_1)'$ and $(\xi_1 \gamma_1)_{-1} = -(s(\gamma_1)\xi_1')_{-1}$. Replacing ξ_1' by $s(\xi)' = \xi - \xi_{-1} X^{-1}$, $\xi \in F$, in the equation $\sum_{i=1}^l \varphi_i(\Delta_{1i}^b(\xi_1')) + \sum_{i=l+1}^n \varphi_i((D\Delta_{1i})^b (\xi_1')) = -(s(\gamma_1)\xi_1')_{-1}$, we arrive at $\sum_{i=1}^l \varphi_i(\Delta_{1i}^b(\xi)) + \sum_{i=l+1}^n \varphi_i((D\Delta_{1i})^b (\xi)) = (\gamma_1^* \xi)_{-1}$ with $\gamma_1^* = -s(\gamma_1) + s(\gamma_1)_0 + \sum_{i=1}^l \varphi_i(\Delta_{1i}^b(X^{-1})) + \sum_{i=l+1}^n \varphi_i((D\Delta_{1i})^b (X^{-1}))$ and ξ running through F.

Now the system (S′) consists of the latter equation and the equations $\sum_{i=1}^{l} \varphi_i(\Delta_{ji}(\xi_j)) + \sum_{i=l+1}^{n} \varphi_i s(D\Delta_{ji}(\xi_j)) = (\gamma_j \xi_j)_{-1}$, $2 \leqslant j \leqslant n$. Note that $\deg \Delta_{11}^{b} = \deg \Delta_{11} - 1$. It is quickly checked that a solution $(\sigma_1, ..., \sigma_n)$ of (S′) yields the solution $(\sigma_1, ..., \sigma_1, -\sigma'_{l+1} + \varphi_{l+1}(1)X^{-1}, ..., -\sigma'_n + \varphi_n(1)X^{-1})$ of (S).

Now we are able to describe the procedure which demonstrates solvability of the system (S). We call (S) trivial if all Δ_{ji} are zero, non-trivial otherwise. Of course we may assume that (S) is non-trivial. In this case we call $d(S) = \min\{\deg \Delta_{ji} \mid 1 \leqslant j, i \leqslant n \text{ and } \Delta_{ji} \neq 0\} \in \{0, 1, 2, ...\}$ the degree of (S), obviously we may assume that $d(S) = \deg \Delta_{11}$. We shall proceed by induction on $n \geqslant 1$ and have to deal with the cases $d(\mathrm{S}) = 0$ and $d(\mathrm{S}) > 0$ separately.

We begin with the easy case $d(\mathrm{S}) = 0$. We may assume that the constant coefficient of Δ_{11} is equal to 1. If $n = 1$, then $\sigma_1 = \beta_1$ is a solution of (S). In case $n > 1$ the first equation yields $\varphi_1(\xi_1) = -\sum_{i=2}^{n} \varphi_i(\Delta_{1i}(\xi_1)) + (\beta_1 \xi_1)_{-1}$. Placing this expression into the remaining equations, we obtain $\sum_{i=2}^{n} \varphi_i((\Delta_{ji} - \Delta_{1i}\Delta_{j1})(\xi_j)) = ((\beta_j - \tilde{\Delta}_{j1}(\beta_1))\xi_j)_{-1}$, $2 \leqslant j \leqslant n$. By hypothesis this system has a solution $(\sigma_2, ..., \sigma_n)$; it is easily verified that $(\beta_1 - \sum_{i=2}^{n} \tilde{\Delta}_{1i}(\sigma_i), \sigma_2, ..., \sigma_n)$ solves (S).

The case $d(\mathrm{S}) > 0$ is more difficult. In certain subcases solvability of (S) is directly shown by use of the induction hypothesis, whereas in the remaining subcases we shall transform (S) into an admissible $n \times n$-system of degree $< d(\mathrm{S})$. After a finite number of steps either (S) is solved by induction, or we arrive at an admissible $n \times n$-system of degree 0. Since the latter is solvable on account of the first part of our proof, (S) is solvable, also.

Subcase I. Each operator Δ_{ji} with $\deg \Delta_{ji} = d(\mathrm{S})$ is regular. In particular Δ_{11} is regular, because $\deg \Delta_{11} = d(S)$ by our assumption. If $n = 1$, then there exists $\sigma_1 \in F$ with $\tilde{\Delta}_{11}(\sigma_1) = \beta_1$, because $\tilde{\Delta}_{11}$ is also regular. The case $n > 1$ likewise falls into two subcases (I′) and (I″):

(I′) $\Delta_{j1} = 0$ for all $2 \leqslant j \leqslant n$. By induction, the system $\sum_{i=2}^{n} \varphi_i(\Delta_{ji}(\xi_j)) = (\beta_j \xi_j)_{-1}$, $2 \leqslant j \leqslant n$, has a solution $\sigma_2, ..., \sigma_n \in F$, i.e., $\sum_{i=2}^{n} \tilde{\Delta}_{ji}(\sigma_i) = \beta_j$ for $2 \leqslant j \leqslant n$. Since $\tilde{\Delta}_{11}$ is regular, there exists $\sigma_1 \in F$ satisfying $\tilde{\Delta}_{11}(\sigma_1) + \sum_{i=2}^{n} \tilde{\Delta}_{1i}(\sigma_i) = \beta_1$. Hence $(\sigma_1, \sigma_2, ..., \sigma_n)$ solves (S).

(I″) $\Delta_{p1} \neq 0$ for some index $2 \leqslant p \leqslant n$. Then $\deg \Delta_{p1} \geqslant \deg \Delta_{11} = d(\mathrm{S})$. Let γ denote the constant coefficient of Δ_{p1}. In case $\gamma \neq 0$ there exists $\alpha \in F$ with $\Delta_{11}(\alpha) = \gamma$. Substituting the operators Δ_{1i}, $1 \leqslant i \leqslant n$, by $\Delta_{1i}\alpha$ and subtracting the first row from the pth one, we obtain an admissible system with an operator Δ'_{p1} at the $(p, 1)$-position, the constant coefficient of which is 0. It is possible that $\Delta'_{p1} = 0$. If $\Delta'_{p1} \neq 0$, then $\deg \Delta'_{p1} \leqslant \deg \Delta_{p1}$; an application of (E) to the pth row yields an admissible $n \times n$-system in which the operator Δ''_{p1} at the $(p, 1)$-position has $\deg \Delta''_{p1} = \deg \Delta'_{p1} - 1$, whereas the operators at the position $(j, 1)$, $j \neq p$, are unchanged. (If $\gamma = 0$, obviously the first part of this step is superfluous.) If $\deg \Delta''_{p1} \geqslant d(S)$, we repeat this procedure; after a finite number of steps we obtain an admissible $n \times n$-system in which the operator at the $(p, 1)$-position either is zero or has degree $< d(\mathrm{S})$. Treating the other rows in an analogous way, if necessary, we arrive after a finite number of steps at an admissible $n \times n$-system which is either of type (I′) or has degree $< d(\mathrm{S})$.

Subcase II. At least one of the operators Δ_{ji} with $\deg \Delta_{ji} = d(\mathrm{S})$ is not regular. Again we may assume that Δ_{11} has this property; i.e., there is $0 \neq \alpha \in F$ such that $\Delta_{11}(\alpha) = 0$. If we substitute the Δ_{1i}, $1 \leqslant i \leqslant n$, by $\Delta_{1i}\alpha$, we obtain an admissible $n \times n$-system in which the constant coefficient of the operator at the $(1, 1)$-position is 0. In the same way as in (I″) an application of (E) to the first row yields an admissible $n \times n$-system of degree $< d(\mathrm{S})$.

At the end we add a "vector-valued" version of Theorem 19 which is needed under Section 2. For that we have to introduce vector-valued differential operators. Let $m \geqslant 1$ be a natural number and $\mathfrak{A}_0, ..., \mathfrak{A}_l \in F^{m \times m}$. Then the k-linear map Δ: $F^m \to F^m$ defined by $(\mathfrak{x})\,\Delta = \sum_{i=0}^{l} \mathfrak{x}^{(i)} \mathfrak{A}_i$ is called a differential operator over F^m, the map $\tilde{\Delta}$: $F^m \to F^m$ given by $\tilde{\Delta}(\mathfrak{x}^t) = \sum_{i=0}^{l} (-1)^i (\mathfrak{A}_i \mathfrak{x}^t)^{(i)}$ the adjoint of Δ. Again we have $(\mathfrak{x} \cdot \tilde{\Delta}(\mathfrak{y}^t))_{-1} = ((\mathfrak{x})\,\Delta \cdot \mathfrak{y}^t)_{-1}$ for all $\mathfrak{x}, \mathfrak{y} \in F^m$. Let $\mathfrak{A}_{i,pq}$ denote the coefficient of $\mathfrak{A}_i$ in the pth row and the qth column and Δ_{pq} the differential operator over F defined by $\Delta_{pq}(\xi) = \sum_{i=0}^{l} \mathfrak{A}_{i,pq} \xi^{(i)}$. Then for $\mathfrak{x} = (\xi_1, ..., \xi_m) \in F^m$ we have $(\mathfrak{x})\,\Delta = (\sum_{p=1}^{m} \Delta_{pq}(\xi_p))_{1 \leqslant q \leqslant m}$ and $\tilde{\Delta}(\mathfrak{x}^t) = (\sum_{q=1}^{m} \tilde{\Delta}_{pq}(\xi_q))^t_{1 \leqslant p \leqslant m}$.

THEOREM 20. *Let be given a family $(\Delta_{ji})_{1 \leqslant i,j \leqslant n}$ of differential operators over F^m and vectors $\mathfrak{b}_1, ..., \mathfrak{b}_n \in F^m$. There exist $\mathfrak{y}_1, ..., \mathfrak{y}_n \in F^m$ with $\sum_{i=1}^{n} \tilde{\Delta}_{ji}(\mathfrak{y}_i^t) = \mathfrak{b}_j^t$, $1 \leqslant j \leqslant n$, if and only if there are linear forms $\varphi_1, ..., \varphi_n \in (F^m, k)$ such that the equations $\sum_{i=1}^{n} \varphi_i((\mathfrak{x}_j)\,\Delta_{ji}) = (\mathfrak{x}_j \mathfrak{b}_j^t)_{-1}$, $1 \leqslant j \leqslant n$, hold for all $\mathfrak{x}_1, ..., \mathfrak{x}_n \in F^m$.*

Proof. One implication being trivial, we suppose that there exist $\varphi_1, ..., \varphi_n \in (F^m, k)$ with $\sum_{i=1}^{n} \varphi_i((\mathfrak{x}_j)\,\Delta_{ji}) = (\mathfrak{x}_j \mathfrak{b}_j)_{-1}$ for all $1 \leqslant j \leqslant n$ and $\mathfrak{x}_1, ..., \mathfrak{x}_n \in F^m$. If we denote by φ_{iq}, $1 \leqslant q \leqslant m$, the restriction of φ_i to the qth coordinate and put $\mathfrak{x}_j = (\xi_{j1}, ..., \xi_{jm})$, then the latter system is equivalent to the system $\sum_{1 \leqslant i \leqslant n, 1 \leqslant q \leqslant m} \varphi_{iq}(\Delta_{ji,pq}(\xi_{jp})) = (\mathfrak{b}_{jp} \xi_{jp})_{-1}$, for $1 \leqslant j \leqslant n$, $1 \leqslant p \leqslant m$. On account of Theorem 19 there are $\sigma_{iq} \in F$ such that $\sum_{1 \leqslant i \leqslant n, 1 \leqslant q \leqslant m} \tilde{\Delta}_{ji,pq}(\sigma_{iq}) = \mathfrak{b}_{jp}$, $1 \leqslant j \leqslant n$, $1 \leqslant p \leqslant n$, hence the vectors $\mathfrak{y}_i = (\sigma_{i1}, ..., \sigma_{im})$, $1 \leqslant i \leqslant m$, satisfy $\sum_{1 \leqslant i \leqslant n} \tilde{\Delta}_{ji}(\mathfrak{y}_i^t) = \mathfrak{b}_j^t$, $1 \leqslant j \leqslant n$.

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